2.4.4 Some more difficult examples

Example
To differentiate

$$\sin\left(\sqrt{1+x^2}\right)$$

we must use the chain rule twice.

$$\begin{aligned}
\frac{d}{dx}\left(\sin\left(\sqrt{1+x^2}\right)\right) &= \frac{d}{dx}\left(\sin\left((1+x^2)^{\frac{1}{2}}\right)\right) \\
&= \cos\left((1+x^2)^{\frac{1}{2}}\right)\cdot\frac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right) \\
&= \cos\left((1+x^2)^{\frac{1}{2}}\right)\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\frac{d}{dx}\left(1+x^2\right) \\
&= \cos\left((1+x^2)^{\frac{1}{2}}\right)\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x \\
&= \frac{x\cos\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}
\end{aligned}$$

Example
To differentiate

$$(x^2+3)\sin x\cos x$$

we must use the product rule twice.

$$\begin{aligned}
\frac{d}{dx}\left((x^2+3)\sin x\cos x\right) &= (x^2+3)\frac{d}{dx}\left(\sin x\cos x\right) + \sin x\cos x\frac{d}{dx}\left(x^2+3\right) \\
&= (x^2+3)\left(\sin x\frac{d}{dx}\left(\cos x\right) + \cos x\frac{d}{dx}\left(\sin x\right)\right) + 2x\sin x\cos x \\
&= (x^2+3)\left(-\sin^2 x + \cos^2 x\right) + 2x\sin x\cos x
\end{aligned}$$

Example
To differentiate

$$\sqrt{x^2-3}\sin x$$

we must use the product rule and the chain rule.

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{x^2-3}\sin x\right) &= \frac{d}{dx}\left((x^2-3)^{\frac{1}{2}}\sin x\right)\\
&= \sin x\frac{d}{dx}\left((x^2-3)^{\frac{1}{2}}\right) + (x^2-3)^{\frac{1}{2}}\frac{d}{dx}(\sin x)\\
&= \sin x\cdot\frac{1}{2}(x^2-3)^{-\frac{1}{2}}\cdot 2x + (x^2-3)^{\frac{1}{2}}\cos x\\
&= \frac{x\sin x}{\sqrt{x^2-3}} + \sqrt{x^2-3}\cos x
\end{aligned}$$

Example
To differentiate

$$\tan x$$

we can use the product rule and the chain rule.

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\
&= \frac{d}{dx}\left(\sin x(\cos x)^{-1}\right)\\
&= \sin x\frac{d}{dx}\left((\cos x)^{-1}\right) + (\cos x)^{-1}\frac{d}{dx}(\sin x)\\
&= \sin x\cdot -(\cos x)^{-2}\cdot -\sin x + (\cos x)^{-1}\cos x\\
&= \frac{\sin^2 x}{\cos^2 x} + \frac{\cos x}{\cos x}\\
&= \tan^2 x + 1 \qquad = \sec^2 x
\end{aligned}$$

This is one way of showing that the derivative of $\tan x$ is $\sec^2 x$.

2.4.5 The quotient rule

The quotient rule is a special case of the product rule. It is up to you whether you learn and use the quotient rule or whether you use the product rule instead.

The quotient rule
If f and g are differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

Proof: Apply the product and chain rules to $f(x)(g(x))^{-1}$

□

Example
To differentiate

$$\tan x$$

we can use the quotient rule.

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\cos x \cos x - \sin x \cdot - \sin x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

This is another way of showing that the derivative of $\tan x$ is $\sec^2 x$.

2.5 Uses of differentiation

2.5.1 Finding the gradient at a point

To find the gradient of a curve at a given x co-ordinate, simply substitute the value of x into the derivative.

Example
To find the gradient of $y(x) = x^3 - x^2$ at $x = 3$, first find $y'(x)$:

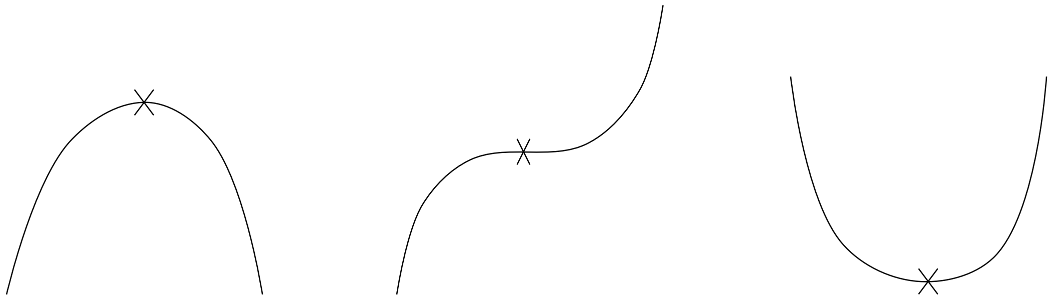
$$y'(x) = 3x^2 - 2x$$

Next subsitute $x = 3$:

$$\begin{aligned}y'(3) &= 3 \cdot 3^2 - 2 \cdot 3 \\ &= 21\end{aligned}$$

2.5.2 Finding the maximum and minimum points

At a point where $\frac{dy}{dx} = 0$, there are three possibilites:



(a) *Local maximum.* (b) *Point of inflection.* (c) *Local minimum.*

Figure 2.6: Different options for when $\frac{dy}{dx} = 0$.

In order to tell which of these occurs at a given point, we must look at the second derivative.

Definition
The **second derivative** of a function, written $f''(x)$ or $\frac{d^2 f}{dx^2}$ is obtained by differentiating $\frac{dy}{dx}$.

Example
If

$$f(x) = x^3$$

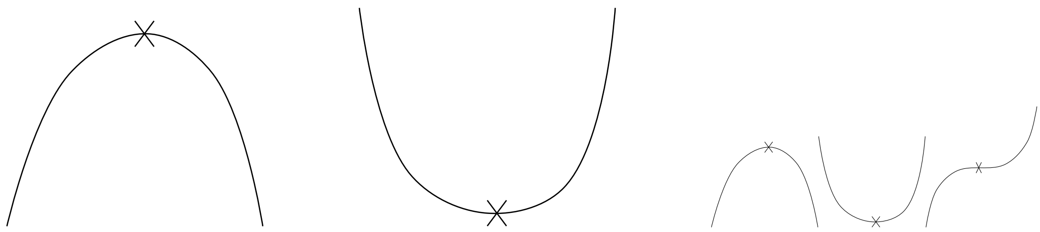
then

$$f'(x) = 3x^2$$

and

$$f''(x) = 6x.$$

The second derivative gives that rate at which the gradient is changing. If the gradient is increasing at a turning point, then the point is a minimum. Similarly, if the gradient is decreasing at a turning point, then the point is a maximum. If the second derivative is 0 at the turning point, we need more information.



(a) Local maximum

$f'(x) = 0$

$f''(x) < 0$

(b) Local minimum

$f'(x) = 0$

$f''(x) > 0$

(c) Need more information

$f'(x) = 0$

$f''(x) = 0$

Figure 2.7: Different options for when $\frac{dy}{dx} = 0$.

Example
Let $f(x) = x^2$.
$f'(x) = 2x$, so f has a turning point at $x = 0$.
$f''(x) = 2$, so $f''(0) > 0$. Therefore the turning point is a minimum

Example
Let $f(x) = x^3$.
$f'(x) = 3x^2$, so f has a turning point at $x = 0$.
$f''(x) = 6x$, so $f''(0) = 0$. We need more information to decide what happens at this point.
In this case, at $x = 0$ there is a point of inflection.

Example
Let $f(x) = x^4$.
$f'(x) = 4x^3$, so f has a turning point at $x = 0$.
$f''(x) = 12x^2$, so $f''(0) = 0$. We need more information to decide what happens at this point.
In this case, at $x = 0$ there is a minimum.