Chapter 2

Differentiation

2.1 Rates of change

Suppose we drive from UCL to Stratford-upon-Avon (100 miles). We plot a graph of the distance travelled against time. We want to measure how fast we traveled. The average speed of the trip is calculated as follows:

$$\frac{100 \text{ miles}}{2 \text{ hrs}} = 50 \text{ mph}.$$

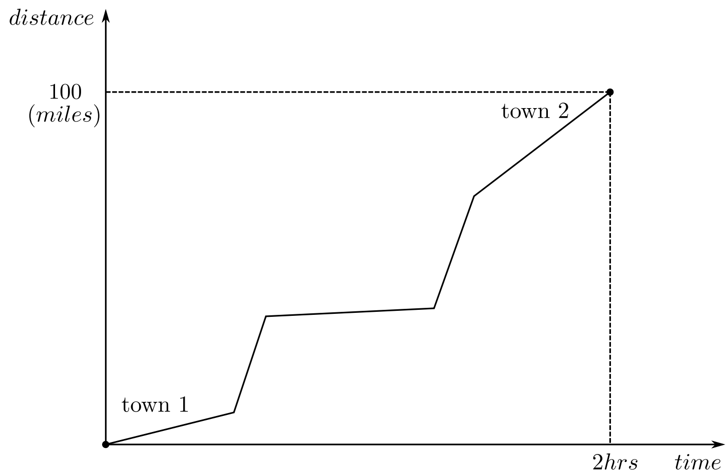

Figure 2.1: Graph showing distance travelled against time, from town 1 (UCL) to town 2 (Stratford-upon-Avon).

However, when travelling you do not stick to one speed, sometimes you do more than 50 mph, sometimes much less. The reading on your speedometer is your *instantaneous* speed. This corresponds to the *gradient* of the graph at the given point in your journey.

Definition
The **gradient** of a line is a measure of the steepness or slope of the line. It can be

found using:

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x}$$

The gradient of a curve is the gradient of the tangent at a given point.

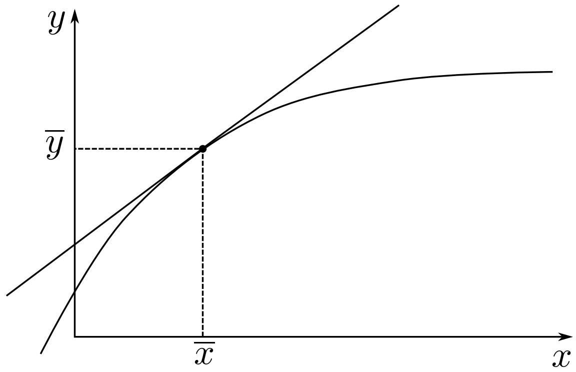

Figure 2.2: Curve $y = f(x)$ with tangent line at $(\overline{x}, \overline{y})$.

In the following section, we will be looking at methods for finding the gradients of graphs.

2.2 Finding the gradient

For mathematical curves, we will learn to find gradients algebraically.

To find the gradient of a curve $y = q(x)$ at $x = c$, we first consider the line joining the points $(c, q(c))$ $(c+h, q(c+h))$, where h is small.

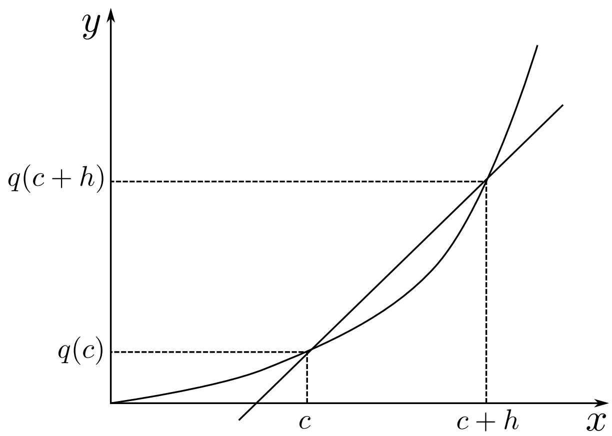

Figure 2.3: Graph showing line joining the points $(c, q(c))$ and $(c+h, q(c+h))$ on the curve $y = q(x)$.

We will look at the gradient of this line as we make h smaller and smaller, as this will get closer and closer to the gradient of the tangent.

Example
Let us start with the example of the curve $y = S(x) = x^2$.

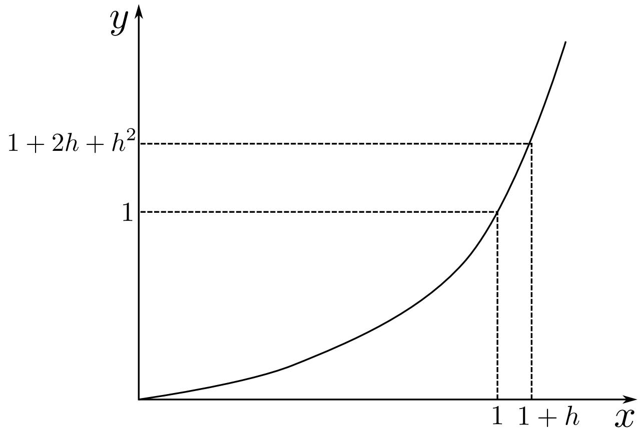

Figure 2.4: Curve $y = S(x) = x^2$ displaying small increment at $x = 1$.

Look at the point $(1, 1)$ on the curve. We want find the gradient at this point. Lets consider a line connecting $(1, 1)$ and $(1 + h, (1 + h)^2)$.
The gradient of this line is:

$$\frac{\text{change in } y}{\text{change in } x} = \frac{(1+h)^2 - 1}{1 + h - 1} \tag{2.1}$$

$$= \frac{h^2 + 2h}{h} \tag{2.2}$$

$$= h + 2 \tag{2.3}$$

To find the gradient at the point, we look at what will happen as $h \to 0$ (h tends to 0).

$$\text{As } h \to 0 \quad h + 2 \to 2$$

Therefore the gradient of the curve $y = x^2$ at the point $x = 1$ is 1.

We define the derivative as follows:

Definition
The **gradient** of $y = f(x)$ at x=c, written $\frac{dy}{dx}$ at c or $f'(c)$, is

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

$\lim_{h\to 0}$ is the limit as h gets closer and closer to 0. This definition is exactly what we used in the example.

If we leave c as a variable instead of subsituting in a value, we can find the gradient of the whole curve.

Example
Let us consider the function $q(x) = x^3$. At $x = c + h$ we have

$$q(c+h) = (c+h)^3 = c^3 + 3c^2h + 3ch^2 + h^3.$$

Therefore,

$$q'(c) = \lim_{h\to 0} \frac{c^3 + 3c^2h + 3ch^2 + h^3 - c^3}{h} \quad (2.4)$$
$$= \lim_{h\to 0} \frac{3c^2h + 3ch^2 + h^3}{h} \quad (2.5)$$
$$= \lim_{h\to 0} 3c^2 + 3ch + h^2 \quad (2.6)$$
$$= 3c^2 \quad (2.7)$$

or in other words,

$$q'(x) = 3x^2.$$

Example
Now let us consider the function $r(x) = 1/x$. In this case we have

$$r(c+h) - r(c) = \frac{1}{c+h} - \frac{1}{c}.$$

Now, let us consider the ratio

$$\frac{r(c+h) - r(c)}{h} = \frac{1}{h}\left(\frac{1}{c+h} - \frac{1}{c}\right) = \frac{1}{h}\left(\frac{-h}{c(c+h)}\right) = -\frac{1}{c(c+h)},$$

and as $h \to 0$, we have

$$r'(c) = -\frac{1}{c^2}, \quad \text{i.e.} \quad r'(x) = -\frac{1}{x^2}, \quad x \neq 0.$$

note: $r(x) = 1/x$ is not well defined at $x = 0$ and in this case, nor is its derivative.