## 1.5 Exponentials

We have seen functions of the form $f(x) = x^a$, where $a$ is a constant. What happens if we swap the $a$ and the $x$ and look at $f(x) = a^x$?

First, let's look at what $a^x$ means for all values of $x \in \mathbb{R}$.

### 1.5.1 Indices

when we wish to multiply a number by itself several times, we make use of index or power notation. We have notation for powers (for $a \in \mathbb{R}$:

$$\begin{aligned} a^2 &= a \cdot a \\ a^3 &= a \cdot a \cdot a \\ a^x &= \overbrace{a \cdot a \ldots a \cdot a}^{x} \quad x \in \mathbb{N} \text{ and } x \neq 0 \end{aligned}$$

Here, $a$ is called the **base** and $x$ is called the **index** or **power**. We also know the following properties:

**Properties of exponents**
For any $a \in \mathbb{R}$ and $x, y \in \mathbb{R}$

1. $a^{x+y} = a^x \cdot a^y$
2. $(a^x)^y = a^{xy}$
3. $a^x \cdot b^x = (ab)^x$

**Examples**

1. $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^3 \cdot 2^2$.
2. $3^6 = 3^{2\times3} = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^2 \cdot 3^2 \cdot 3^2 = (3^2)^3$.
3. $2^3 \cdot 3^3 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = (2 \cdot 3)^3$.

We can use the properties of exponents to justify the definitions of $a^x$ when $x$ is not a positive integer. Throughout this we will assume that $a > 0$.

**For** $x \in \mathbb{Z}$

For $x = 0$, we notice that:

$$\begin{aligned} a^2 &= a^{2+0} \\ &= a^2 \cdot a^0 \end{aligned}$$

This shows that:

$$a^0 = 1$$

When $x$ is a negative integer:

$$\begin{aligned} 1 &= a^0 \\ &= a^{x-x} \\ &= a^x \cdot a^{-x} \end{aligned}$$

Dividing by $a^x$, we get:

$$a^{-x} = \frac{1}{a^x}$$

**For $x \in \mathbb{Q}$ (the set of all fractions)**

For $x = \frac{1}{n}$, notice that:

$$\begin{aligned} \left(a^{\frac{1}{n}}\right)^n &= a^{\frac{1}{n} \cdot n} \\ &= a^1 \\ &= a \end{aligned}$$

Taking the $n$th root gives:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Similarly, we find that

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

**For $x \in \mathbb{R}$**

If $x$ is an irrational number, then, for any small number $\epsilon > 0$ we can always find two rational numbers $c$ and $d$ which satisfy $x - \epsilon < c < x < d < x + \epsilon$. $a^x$ is defined to be the limit of $a^c$ (or $a^d$) as $\epsilon \to 0$.

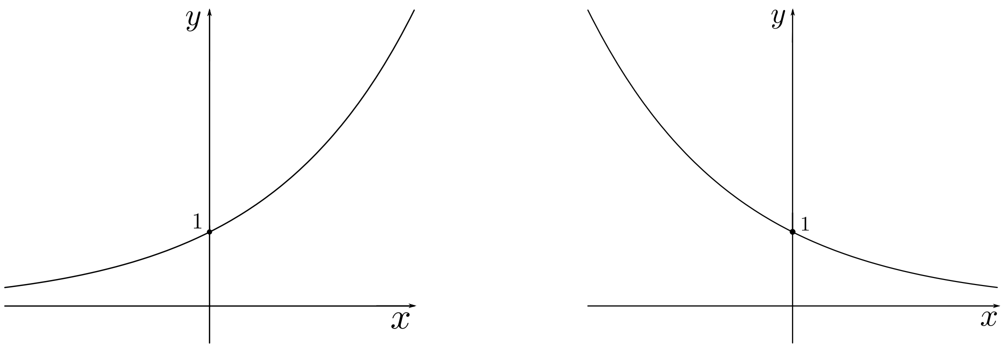
Finally, an exponential function can be defined by

$$f(x) = a^x, \quad x \in \mathbb{R},$$

where $a$ is a positive constant. The domain of $f$ is $\mathbb{R}$ and the range is $\mathbb{R}^+$.

If $a < 1$, it is common to define $b = \frac{1}{a}$. $f$ can then be written as $f(x) = b^{-x}$ with $b > 1$.

The graph of f is as follows:




(a) $y = a^x, \quad a > 1.$ (b) $y = a^{-x}, \quad a > 1.$

Figure 1.5: Comparison of exponent graphs for different values of $a$.

There is also a special exponential function, $f = e^x$, we will investigate this further later in the course.