1.4.5 Degree ≥ 3 polynomials

In general, we have the algebraic equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0, \tag{1.10}$$

which has n roots, including real and complex roots.

$n = 2$	we have formulae for roots	(quadratics)
$n = 3$	we have formulae for roots	(cubic)
$n = 4$	we have formulae for roots	(quartics)
$n > 4$	No general formulae exist	(proven by Évariste Galois)

But in any case, we may try factorisation to find the roots. We have the useful theorem:

Theorem: Factor theorem
Let P be a polynomial of degree n. For $a \in \mathbb{R}$ (or C),

$$P(a) = 0 \quad \text{if and only if} \quad P(x) = (x - a)Q(x)$$

where Q is a polynomial of degree $n - 1$.

Once one root is found, this theorem can be used to factorise the polynomial.

Example
Consider $P(x) = x^3 - 8x^2 + 19x - 12$. We know that $x = 1$ is a solution to $P(x) = 0$, then it can be shown that

$$P(x) = (x - 1)Q(x) = (x - 1)(x^2 - 7x + 12).$$

Here $P(x)$ is a cubic and thus $Q(x)$ is a quadratic.

The next examples show two methods of finding $Q(x)$.

Example: Comparing coefficients
Consider $P(x) = x^3 - x^2 - 3x - 1$. By observation, we know

$$P(-1) = (-1)^3 - (-1)^2 - 3(-1) - 1 = 0.$$

So $x_1 = -1$ is a root. Let us write

$$P(x) = (x + 1)(ax^2 + bx + c),$$

then multiplying the brackets we have

$$P(x) = ax^3 + (a + b)x^2 + (b + c)x + c,$$

which should be equivalent to $x^3 - x^2 - 3x - 1$. Thus, comparing the corresponding coefficients we have

$$\begin{aligned} a &= 1, \\ a + b &= -1, \\ b + c &= -3, \\ c &= -1. \end{aligned}$$

This set of simultaneous equations has the solution

$$a = 1, \quad b = -2, \quad c = -1.$$

So we can write

$$P(x) = (x + 1)(x^2 - 2x - 1)$$

To find the other two solutions of $P(x) = 0$, we must set $(x^2 - 2x - 1) = 0$ which has solutions $x_{2,3} = 1 \pm \sqrt{2}$, together with $x_1 = -1$ we have a complete set of solutions for $P(x) = 0$.

As with many areas of mathematics, there are many ways to tackle a problem. Another way to find $q(x)$ given you know some factor of $P(x)$, is called *polynomial division*.

Example: Long division of polynomials
Consider $P(x) = x^3 - x^2 - 3x - 1$, we know $P(-1) = 0$. The idea is that we "divide" $P(x)$ by the factor $(x + 1)$, like so:

$$\begin{array}{rl} & x^2 - 2x - 1 \\ \hline x + 1 \big) & x^3 - x^2 - 3x - 1 \\ & -x^3 - x^2 \\ \hline & -2x^2 - 3x \\ & 2x^2 + 2x \\ \hline & -x - 1 \\ & x + 1 \\ \hline & 0 \end{array}$$

Hence, multiplying the *quotient* by the *divisor* we have $(x + 1)(x^2 - 2x - 1) = x^3 - x^2 - 3x - 1$.