4.4.2 Finding a particular integral

The particular integral depends on the function $p(x)$. We only consider three categories of $p(x)$:

1. polynomials
2. trigonometric functions
3. exponential functions

p is a polynomial

When p is a polynomial, we guess that the particular integral will be a polynomial of the same order.

Example
Find the general solution to the differential equation

$$y'' + 2y' + y = x^2.$$

Recall, the general solution takes the form $y = f(x) + g(x)$. Using the method in the previous section, we know that the C.F. is

$$g(x) = c_1e^{-x} + c_2xe^{-x}$$

or

$$g(x) = (c_1 + c_2x)e^{-x}.$$

Next, we must find the particular integral (P.I.), we try

$$f(x) = ax^2 + bx + x.$$

We find

$$f'(x) = 2ax + b, \quad f''(x) = 2a.$$

Substituting into the differential equation gives

$$\begin{aligned} f'' + 2f' + f &= 2a + 2(2ax + b) + ax^2 + bx + c \\ &= ax^2 + (4a + b)x + 2a + 2b + c \\ &\equiv x^2. \end{aligned}$$

Comparing coefficients between the LHS and the RHS we have

$$\left.\begin{array}{l} a = 1 \\ 4a + b = 0 \\ 2a + 2b + c = 0 \end{array}\right\} \implies \left.\begin{array}{l} a = 1 \\ b = -4 \\ c = 6 \end{array}\right\} \implies f(x) = x^2 - 4x + 6,$$

Finally, we can write the general solution as

$$y(x) = x^2 - 4x + 6 + (c_1 + c_2x)e^{-x}.$$

p is a trigonometric function

If p is a sin or cos, we guess that the particular integral will involve sin and cos.

Example
Solve the following initial-value problem:

$$y'' - 2y' + y = \sin x, \quad y(0) = -2, \quad y'(0) = 2.$$

[Notice that we have two boundary conditions here because second order differential equations have two constants of integration to be found.]
The C.F. for this problem is

$$g(x) = (c_1 + c_2 x)e^x.$$

To find the P.I. we try

$$f = a\sin x + b\cos x.$$

We find

$$f' = a\cos x - b\sin x, \quad f'' = -a\sin x - b\cos x.$$

Substituting into the differential equation we have

$$\begin{aligned} f'' - 2f' + f &= -a\sin x - b\cos x - 2a\cos x + 2b\sin x + a\sin x + b\cos x \\ &= (-a + 2b + a)\sin x + (-b - 2a + b)\cos x \\ &= 2b\sin x - 2a\cos x \\ &\equiv \sin x. \end{aligned}$$

Comparing coefficients, we have

$$a = 0, \quad b = \frac{1}{2} \quad \Longrightarrow \quad f = \frac{1}{2}\cos x.$$

Therefore the general solution to the initial-value problem is

$$y(x) = \frac{1}{2}\cos x + (c_1 + c_2 x)e^x.$$

In order to find the unknown constants c_1 and c_2 using the boundary conditions, we need to find $y'(x)$, so we differentiate the above to give:

$$\begin{aligned} y'(x) &= -\frac{1}{2}\sin x + c_2 e^x + (c_1 + c_2 x)e^x \\ &= -\frac{1}{2}\sin x + (c_1 + c_2 + c_2 x)e^x \end{aligned}$$

The boundary conditions give:

$$\begin{aligned} y(0) &= \frac{1}{2}\cos 0 + e^0(c_1 + c_2 \cdot 0) \\ &= \frac{1}{2} + c_1 = -2 \\ y'(0) &= -\frac{1}{2}\sin 0 + e^0(c_1 + c_2 + c_2 \cdot 0) \\ &= c_1 + c_2 = 2 \end{aligned}$$

Thus, we have the constants

$$c_1 = -\frac{5}{2}, \quad c_2 = 2 - c_1 = 2 + 52 = \frac{9}{2}.$$

Finally, the solution to the initial value problem is

$$y(x) = \frac{1}{2}\cos x + \frac{1}{2}e^x(9x - 5).$$