3.6 Numerical integration

Consider evaluating the definite integral

$$\int_a^b f(x)\,dx.$$

For the vast majority of function, the antiderivataive is not known. For example, we can't find:

$$\int \sqrt{1+x^3}\,dx \quad \text{or} \quad \int e^{x^2}\,dx.$$

When applying integration to a real application this is a problem.

When an impossible integral is encountered, we must use a **numerical method** to approximate the answer.

3.6.1 Trapezium method

We want to estimate the integral of $f(x)$ on the interval $[a, b]$, which represents the area under the curve $y = f(x)$ from a to b.

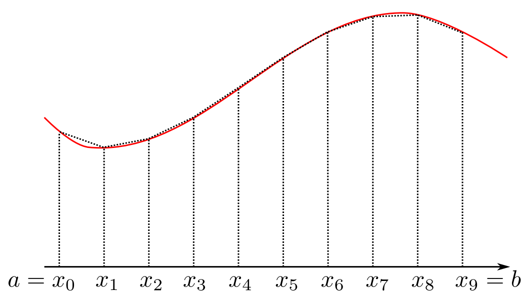

Figure 3.6: Forming trapeziums with height of the sides dictated by the curve $y = f(x)$ over the interval $[a, b]$.

We choose n number of pieces. Divide the interval $a \le x \le b$ into n (equal) pieces with points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

On each piece of the interval, we build a trapezium by joining points on the curve by a straight line. We calculate the total area by summing all the area of the trapezia. This is our estimate of the integral.

To start with, let h be the width of one piece of the interval, i.e.

$$h = \frac{b-a}{n},$$

then we have

$$x_k = x_0 + kh, \quad k = 0, 1, 2, \ldots, n. \quad x_0 = a, \quad x_n = b.$$

Let us consider the trapezium based on the piece $[x_{k-1}, x_k]$, whose width is h. The height of the sides of the trapezium are $f(x_{k-1})$ and $f(x_k)$. So the area is

$$h\frac{f(x_{k-1}) + f(x_k)}{2}.$$

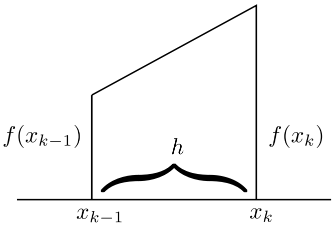

Figure 3.7: *Trapezium constructed over each piece of the interval, where each piece has width h.*

Then the total area under the curve over $[a, b]$ is the sum:

$$\begin{aligned}
\text{Area} &= h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2} \\
&= \frac{h}{2}\left[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\right] \\
&= \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + \cdots + f(x_{n-1})) + f(x_n)\right] \\
&= \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right].
\end{aligned}$$

We can think of the sum as follows, we have the two outer sides of the first and last trapezium, then every trapezium in-between shares its sides with its neighbour, therefore we require two lots of the interior sides.

Example
Using the trapezium method, estimate

$$\int_0^1 \frac{1}{1 + x^4}\, dx.$$

We choose $n = 4$, then

$$h = \frac{1 - 0}{4} = \frac{1}{4}, \quad x_k = kh, \quad k = 0, 1, 2, 3, 4.$$

Also, note that

$$f(x) = \frac{1}{1 + x^4}, \quad \text{i.e.} \quad f(x_k) = \frac{1}{1 + x_k^4}.$$

Therefore, we have

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^4}\,dx &\approx \frac{h}{2}\left[f(x_0)+2(f(x_1)+f(x_2)+f(x_3))+f(x_4)\right] \\
&= \frac{1}{8}\left[f(0)+2(f\left(\frac{1}{4}\right)+f\left(\frac{1}{2}\right)+f\left(\frac{3}{4}\right))+f(1)\right] \\
&= \frac{1}{8}\left[1+2\left(\frac{256}{257}+\frac{16}{17}+\frac{256}{337}\right)+\frac{1}{2}\right] \\
&= 0.862.
\end{aligned}$$

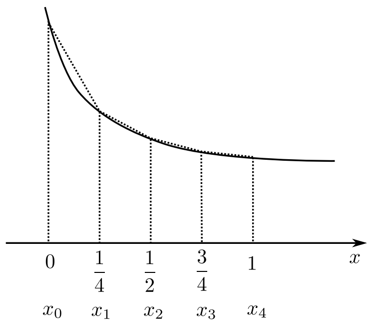

Figure 3.8: Numerically integrating under $y = 1/(1+x^4)$. Dividing interval into 4 pieces of width $h = 1/4$.

This is an over-estimate of the integral since $y = f(x)$ is convex (i.e. it curves up like a cup). If it were concave (i.e. curved down like a cap), then you would have an under-estimate.