3.3.5 Integration by parts

This is equivalent to the product rule for integration. Suppose we have two function $u(x)$ and $v(x)$. Then the product rule states

$$\frac{d}{dx}(uv) = u'v + uv'.$$

Rearranging the above gives

$$uv' = \frac{d}{dx}(uv) - u'v.$$

Integrating both sides we get

$$\begin{aligned}\int uv' \; dx &= \int \frac{d}{dx}(uv) \; dx - \int u'v \; dx \\ &= uv - \int u'v \; dx.\end{aligned}$$

So we write the rule for integration by parts as:

Integration by Parts

$$\int u\frac{dv}{dx} \; dx = uv - \int v\frac{du}{dx} \; dx.$$

Example

$$\int xe^x \; dx$$

Let:

$$u = x, \quad v' = e^x$$

This means that:

$$u' = 1, \quad v = e^x.$$

Therefore, we can calculate the integral as follows:

$$\begin{aligned}\int xe^x \; dx &= \int uv' \; dx \\ &= uv - \int u'v \; dx \\ &= xe^x - \int e^x \; dx \\ &= xe^x - e^x + c.\end{aligned}$$

Check:

$$\frac{d}{dx}\left[xe^x - e^x + c\right] = e^x + xe^x - e^x = xe^x.$$

Example

$$\int \ln x \; dx$$

This is the same as

$$\int 1 \cdot \ln x \; dx$$

We choose:

$$u = \ln x, \quad v' = 1$$

This means that:

$$u' = \frac{1}{x}, \quad v = x.$$

So we calculate the integral as

$$\begin{aligned} \int \ln x \; dx &= \int 1 \cdot \ln x \; dx \\ &= \int uv' \; dx \\ &= uv - \int u'v \; dx \\ &= x \ln x - \int \frac{1}{x} \cdot x \; dx \\ &= x \ln x - x + c. \end{aligned}$$

Check:

$$\frac{d}{dx}\left[x \ln x - x + c\right] = \ln x + x \cdot \frac{1}{x} - 1 = \ln x.$$

Example

$$\int e^x \cos x \; dx$$

Let:

$$u = \cos x, \quad v' = e^x$$

This means that

$$u' = -\sin x, \quad v = e^x$$

So we write our integral as

$$\begin{aligned} \int e^x \cos x \; dx &= \int uv' \; dx \\ &= uv - \int u'v \; dx \\ &= e^x \cos x + \int e^x \sin x \; dx. \end{aligned}$$

Now, we have an integral similar to what we started with, so let us integrate this by parts too, choosing

$$\bar{u} = \sin x, \quad \bar{v}' = e^x \qquad\qquad \bar{u}' = \cos x, \quad \bar{v} = e^x.$$

So our original integral becomes

$$\begin{aligned}\int e^x \cos x \; dx &= e^x \cos x + \int \bar{u}\bar{v}' \; dx \\ &= e^x \cos x + \bar{u}\bar{v} - \int \bar{u}'\bar{v} \; dx \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x \; dx.\end{aligned}$$

Note, now on the RHS we have the same integral we started with. Rearranging this, we can make the integral the subject:

$$\int e^x \cos x \; dx = e^x \cos x + e^x \sin x - \int e^x \cos x \; dx,$$

$$\therefore \quad 2\int e^x \cos x \; dx = e^x \cos x + e^x \sin x.$$

So finally, we can write:

$$\int e^x \cos x \; dx = \frac{1}{2}\left[e^x\left(\cos x + \sin x\right)\right] + c,$$

Check:

$$\frac{d}{dx}\left[\frac{1}{2}\left[e^x\left(\cos x + \sin x\right)\right] + c\right] = \frac{1}{2}\left\{e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)\right\} = e^x \cos x$$