Chapter 3

Integration

3.1 The basic idea

If we have a distance-time graph, the gradient of the graph gives us the velocity at that point. In the previous chapter, we learnt how to find the gradient at a point on a curve.

If we have a velocity-time graph, the area under the curve gives us the distance travelled. In this chapter, we will learn how to find this.

The method of finding the area under a curve is called **integration**. We will write the area under a curve $y = f(x)$ between $x = a$ and $x = b$ as

$$\int_a^b f(x)\ dx.$$

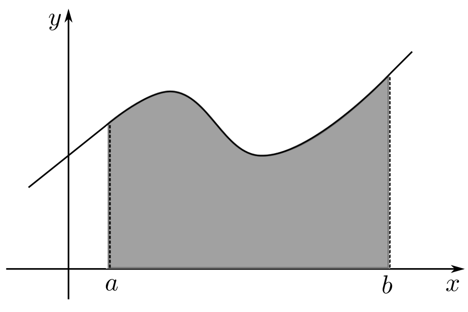


Figure 3.1: $\int_a^b f(x)\ dx$.

3.1.1 Finding the area under a curve

To find the area under a curve, we begin in a similar way to how we began differentiation:

1. We divide the interval $a \leq x \leq b$ into small pieces, each of length h.
2. We build a rectangle on each piece, where the top touches the curve.

3. We calculate the total area of the rectangles.

As we make h get smaller and smaller, the area of the rectangles gets closer and closer to the area under the curve.

Example

Consider the function $f(x) = x$ on the interval $0 \le x \le 1$.
We divide $[0, 1]$ into n equal pieces, each of width $h = \frac{1}{n}$. The divisions occur at

$$0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k-1}{n}, \frac{k}{n}, \ldots, \frac{n-1}{n}, 1$$

or

$$0, h, 2h, \ldots, n-1)h, 1$$

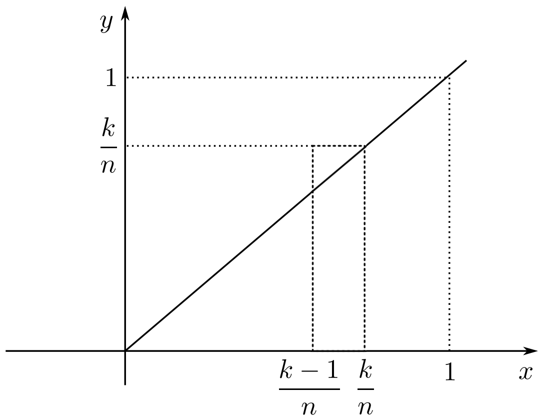


Figure 3.2: *The rectangle between $(k-1)h$ and kh.*

The rectangle between $(k-1)h$ and kh will have height $f(kh) = kh$, and the area of this rectangle is

$$\underbrace{kh}_{\text{height}} \cdot \underbrace{h}_{\text{width}} = kh^2.$$

The sum of the area of all rectangles on the interval is

$$\begin{aligned} h^2 + 2h^2 + \cdots + nh^2 &= h^2(1 + 2 + \cdots + n) \\ &= h^2 \frac{n(n+1)}{2} \\ &= h^2 \frac{\frac{1}{h}(\frac{1}{h} + 1)}{2} \\ &= \frac{1+h}{2}. \end{aligned}$$

As $h \to 0$, $\frac{1+h}{2} \to \frac{1}{2}$. Therefore,

$$\int_0^1 x \, dx = \frac{1}{2}.$$

As we did with differentiation, we would like to find faster methods of finding the area under a curve. To do this, we relate integration and differentiation.

3.1.2 The fundamental theorem of calculus

It is often treated as obvious that integration and differentiation are opposites. However, it is un-obvious enough that mathematicians have a big theorem about it:

Theorem: Fundamental Theorem of Calculus

$$\int_a^b g'(x)\ dx = g(b) - g(a)$$

In other words, if we are trying to find

$$\int_a^b f(x)\ dx$$

then if we can find a function $F(x)$ so that $F'(x) = f(x)$,

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Definition
We call $F(x)$ the **antiderivative** of $f(x)$.

The aim of this chapter is to learn methods for finding the antiderivative.

3.1.3 Indefinite and definite integrals

Let $f(x)$ be a function. If $F(x)$ is the antiderivative of $f(x)$, then $F(x) + 4$ is also the antiderivative of $f(x)$.

Proof:

$$\begin{aligned}\frac{d}{dx}(F(x) + 4) &= \frac{d}{dx}(F(x)) + \frac{d}{dx}(4)\\ &= f(x) + 0\end{aligned}$$

□

Similarly, $F(x) + c$ will be the antiderivative of $f(x)$ for any $c \in \mathbb{R}$. When finding an integral without limits, we must include this constant term.

Definition

The **indefinite integral** of a function $f(x)$ is

$$\int f(x)\ dx = F(x) + c$$

where $F(x)$ is any derivative of $f(x)$. Ususally, we pick $F(x)$ as the antiderivative without a constant term.

Definition

$$\int_a^b f(x)\ dx = F(x)$$

is called the **definite integral**.

3.2 Finding integrals

3.2.1 Polymonials and other powers

To find indefinite integrals, we are going to look for functions which will have the correct derivative.

$$\int ax^b\ dx = \frac{ax^{b+1}}{b+1} + c$$

Proof:

$$\frac{d}{dx}\left(ax^B\right) = aBx^{B-1}$$

so

$$\frac{d}{dx}\left(\frac{ax^B}{B}\right) = ax^{B-1}.$$

Replacing B with $b+1$ gives the correct result.

□

Example

$$\int x^2\ dx = \frac{x^3}{3} + c$$

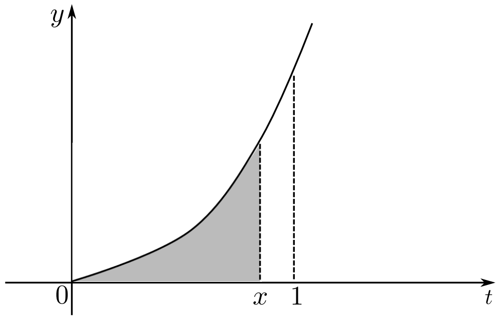
Example

$$\int x^2 + x^3 \; dx = \frac{x^3}{3} + \frac{x^4}{4} + c$$

Example
To find

$$\int_0^x t^2 \; dt,$$

or the area under $y = t^2$ between $t = 0$ and $t = x$:



We first find

$$\int t^2 \; dt = \frac{t^3}{3} + c.$$

Then

$$\begin{aligned}\int_0^x t^2 \; dt &= \frac{x^3}{3} + c - \frac{0^3}{3} - c \\ &= \frac{x^3}{3}.\end{aligned}$$

For $x = 1$, the area under $y = t^2$ between $t = 0$ and $t = 1$ is

$$\begin{aligned}\int_0^1 t^2 \; dt &= \frac{1^3}{3} \\ &= \frac{1}{3}.\end{aligned}$$

When finding definite integrals, the constants will always cancel, so can be ignored. We often write the working like this, with the indefinite integral in brackets:

Example

$$\begin{aligned}\int_1^3 3x^2 \; dx &= \left[x^3\right]_1^3 \\ &= 3^3 - 1^3 \\ &= 26\end{aligned}$$

This also works with negative and fractional powers.

Example
Suppose we want to integrate the function $1/x^2$ over the interval $[1, 2]$. That is, we want to calculate

$$\int_1^2 \frac{1}{x^2}\ dx.$$

If we put $F(x) = -1/x$, then

$$F'(x) = \frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}.$$

So we can write

$$\int_1^2 \frac{1}{x^2}\ dx = \left[-\frac{1}{x}\right]_1^2 = \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right) = \frac{1}{2}.$$

The integral $\int_1^2 \frac{1}{x^2}\ dx$ represents the area under the curve $y = \frac{1}{x^2}$ between 1 and 2, therefore we understand that this integral makes some geometrical sense.

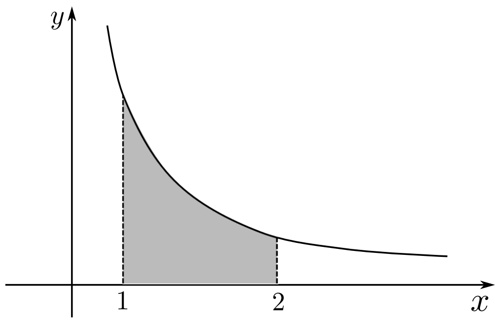


Figure 3.3: Integrating to find the shaded area under the curve $y = \frac{1}{x^2}$ on the interval $[1, 2]$.

Example

$$\begin{aligned}\int \sqrt{x}\ dx &= \int x^{\frac{1}{2}}\ dx \\ &= 2x^{\frac{3}{2}} + c\end{aligned}$$

x^{-1} is a special case, as using the same rule would require division by 0:

$$\int \frac{1}{x}\ dx = \ln|x| + c$$

Proof: This is true because for $x > 0$

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

and for $x < 0$,

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}.$$

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

□

3.2.2 Exponential functions

$$\int e^x \; dx = e^x + c$$

Proof: This is true because

$$\frac{d}{dx}e^x = e^x.$$

□

$$\int a^x \; dx = \frac{a^x}{\ln a} + c$$

Proof: This is true because

$$\begin{aligned}\frac{d}{dx}\left(\frac{a^x}{\ln a}\right) &= \frac{1}{\ln a}\frac{d}{dx}\left(e^{x \ln a}\right) \\ &= \frac{1}{\ln a}e^{x \ln a}\ln a \\ &= e^{x \ln a}.\end{aligned}$$

□

3.2.3 Trigonometric functions

$$\int \cos x \; dx = \sin x + c, \qquad \text{since } \frac{d}{dx}(\sin x) = \cos x,$$

$$\int \sin x \; dx = -\cos x + c, \qquad \text{since } \frac{d}{dx}(-\cos x) = \sin x.$$

3.3 Rules for integration

Instead of making a longer and longer list of functions and their antiderivatives, we are going to learn some rules for integration and use these to work out harder integrals

3.3.1 Sum rule and constants

Sum Rule

$$\int (f(x) + g(x)) \; dx = \int f(x) \; dx + \int g(x) \; dx \tag{3.1}$$

Multiplication by a constant

$$\int Kf(x) \; dx = K \int f(x) \; dx \tag{3.2}$$

Both these rules follow from the equivalent rules differentiation.