2.7 Differentiating inverse functions

Definition
If a function f is one-to-one, we can find its **inverse**, f^{-1}. The inverse satisfies

$$f^{-1}(f(x)) = x$$

for all values of x in the domain of f.

This notation is most commonly used for trigonometric functions ($\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.).

note: $\tan^{-1} x$ is used for the inverse of tan and NOT $\frac{1}{\tan x}$.

note: Sometimes, arcsin, arccos and arctan are used to represent $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Finding the derivative of an inverse
Let f be a function. The derivative of f^{-1} is:

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

Proof: Let f be a function and let $g = f^{-1}$. By the chain rule,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

g is f^{-1}, so $f(g(x)) = x$ and

$$\frac{d}{dx} f(g(x)) = 1.$$

Therefore,

$$1 = f'(g(x))g'(x).$$

Rearranging gives

$$g'(x) = \frac{1}{f'(g(x))}.$$

□

The method in the proof can be used to differentiate inverse functions:

Example
To find

$$\frac{d}{dx}\left(\sin^{-1} x\right),$$

we first look at

$$\frac{d}{dx}\left(\sin\left(\sin^{-1}x\right)\right) = \frac{d}{dx}\left(x\right)$$
$$= 1.$$

Using the chain rule,

$$\frac{d}{dx}\left(\sin\left(\sin^{-1}x\right)\right) = \cos\left(\sin^{-1}x\right)\cdot\frac{d}{dx}\left(\sin^{-1}x\right).$$

Therefore:

$$\cos\left(\sin^{-1}x\right)\cdot\frac{d}{dx}\left(\sin^{-1}x\right) = 1$$
$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\cos\left(\sin^{-1}x\right)}$$

We can simplify this, by letting $\theta = \sin^{-1}x$, then looking at the following triangle:

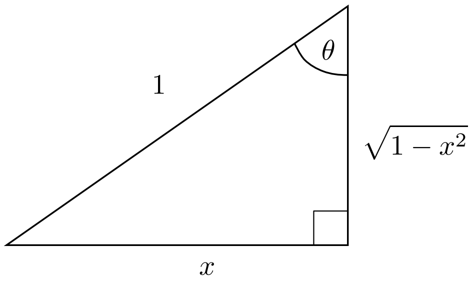

Figure 2.11: $\sin(\theta) = x$

This tells us that $\cos\theta = \sqrt{1-x^2}$. Therefore

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}.$$

We can also find the derivatives of inverses by using:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Example
To find

$$\frac{d}{dx}\left(\sin^{-1}x\right),$$

let $y = \sin^{-1}x$. This means that $x = \sin y$ and so:

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$
$$= \frac{1}{\cos y}$$
$$= \frac{1}{\cos\left(\sin^{-1} x\right)}$$

Simplifying as before gives

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}.$$

Example

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sin(\cos^{-1} x)}$$
$$= -\frac{1}{\sqrt{1-x^2}}$$

Example

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+\tan^2(\tan^{-1} x)}$$
$$= \frac{1}{1+x^2}$$

2.7.1 Differentiating Logarithms

Property

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example
To find

$$\frac{d}{dx}\left(\ln(\cos x)\right)$$

we must use the chain rule. Choose $g(x) = \cos x$ and $f(u) = \ln u$, so we have

$g'(x) = -\sin x$ and $f'(u) = 1/u$. Thus

$$\begin{aligned} \frac{d}{dx}(\ln(\cos x)) &= f'(g(x))g'(x) \\ &= \frac{1}{\cos x} \cdot (-\sin x) \\ &= -\tan x. \end{aligned}$$

Example

$$\begin{aligned} \frac{d}{dx}(\sin(\ln x)) &= \cos(\ln x) \cdot \frac{1}{x} \\ &= \frac{\cos(\ln x)}{x}. \end{aligned}$$

Property: Change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular,

$$\log_a x = \frac{\ln x}{\ln a}.$$

Proof: Let

$$m = \log_a x.$$

This means that

$$a^m = x.$$

Applying $\log_b$ to both sides gives:

$$\log_b a^m = \log_b x$$

$$m \log_b a = \log_b x$$

$$m = \frac{\log_b x}{\log_b a}$$

□

Property

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

Proof:

$$\begin{aligned}\frac{d}{dx}\left(\log_a x\right) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right)\\ &= \frac{1}{x\ln a}\end{aligned}$$

□