Definition
If the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists, the function f is **differentiable**.

Example
An example of a function which is not differentiable at a certain point:

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} .$$

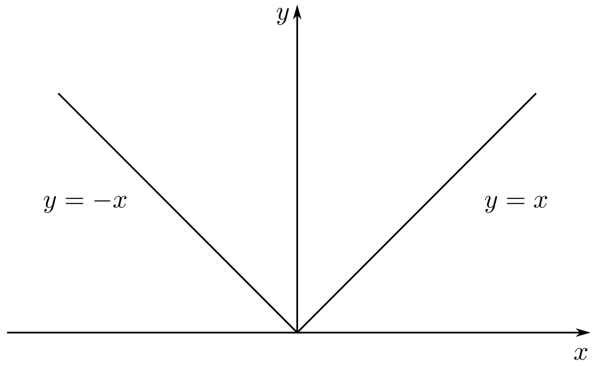

Figure 2.5: *Graph of* $y = |x|$.

At $x = 0$, $f(x)$ is continuous but not differentiable, since through the point $(0, 0)$, you can draw many, many tangent lines. We can also show

$$\text{for} \quad h > 0, \quad \frac{f(0+h)-f(0)}{h} = \frac{h-0}{h} = 1,$$

$$\text{for} \quad h < 0, \quad \frac{f(0+h)-f(0)}{h} = \frac{-h-0}{h} = -1,$$

i.e.

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

doesn't exist! Taking the limit from both sides must give the same answer.

2.3 Some common derivatives

The derivative of x^n is nx^{n-1}.

Proof:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{\{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n\} - x^n}{h} \\
&= \lim_{h\to 0}\left\{nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right\} \\
&= nx^{n-1}.
\end{aligned}$$

□

The derivative of $\sin x$ is $\cos x$.

note: For this to be true, x must be measured in radians.

Proof:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\
&\quad \text{When } h \text{ is small, } \sin h \approx h \text{ and } \cos h \approx 1\text{, so:} \\
&= \lim_{h\to 0}\frac{\sin x + h\cos x - \sin x}{h} \\
&= \lim_{h\to 0}\cos x \\
&= \cos x
\end{aligned}$$

□

The derivative of $\cos x$ is $-\sin x$.

note: For this to be true, x must be measured in radians.

Proof:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x-h\sin x-\cos x}{h} \\
&= \lim_{h\to 0}-\sin x \\
&= -\sin x
\end{aligned}$$

□

The derivative of $\tan x$ is $\sec^2 x$.
We will see why this is true later.

We could continue working through all the functions we would like to differentiate and working out their derivatives, but this take a long time. Instead, there are some rules which we can use to save time.

2.4 Rules for differentiation

There are three key rules we can use to differentiate more complicated functions.

2.4.1 The sum rule

The sum rule
If f and g are differentiable, then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

or

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

Example

Consider the function $f(x) = \left(x^3 + x^4\right)$, then using the above we have

$$\begin{aligned}\frac{d}{dx}\left(x^3 + x^4\right) &= \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^4\right)\\ &= 3x^2 + 4x^3.\end{aligned}$$

If you repeatedly apply the sum rule, you have

$$\frac{d}{dx}\left(f_1(x) + f_2(x) + \cdots + f_n(x)\right) = \frac{d}{dx}\left(f_1(x)\right) + \frac{d}{dx}\left(f_2(x)\right) + \cdots + \frac{d}{dx}\left(f_n(x)\right).$$

2.4.2 The product rule

The product rule
If f and g are differentiable, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}\left(f(x)\right)g(x) + f(x)\frac{d}{dx}\left(g(x)\right)$$

or

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Example

$$\begin{aligned}\frac{d}{dx}\left[(x^2+1)(x^3-1)\right] &= 2x(x^3-1) + (x^2+1)3x^2\\ &= 2x^4 - 2x + 3x^4 + 3x^2\\ &= 5x^4 + 3x^2 - 2x.\end{aligned}$$

Here we have put

$$f(x) = x^2 + 1 \quad \Longrightarrow \quad f'(x) = 2x,$$

and

$$g(x) = x^3 - 1 \quad \Longrightarrow \quad g'(x) = 3x^2.$$

Example
Consider the derivative of x^5, so

$$\begin{aligned}\frac{d}{dx}\left(x^5\right) &= \frac{d}{dx}\left(x^4 \cdot x\right)\\ &= 4x^3 \cdot x + x^4 \cdot 1\\ &= 5x^4,\end{aligned}$$

as expected, since

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

Here we have put

$$f(x) = x^4 \quad \Longrightarrow \quad f'(x) = 4x^3,$$

and

$$g(x) = x \quad \Longrightarrow \quad g'(x) = 1.$$

2.4.3 The chain rule

The chain rule
If f and g are differentiable, then

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$$

Example
Consider the function $y(x) = (x^3 + 2x)^{10}$. Here we will choose $f(w) = w^{10}$ and $g(x) = x^3 + 2x$, (so $f'(w) = 10w^9$ and $g'(x) = 3x^2 + 2$). Then

$$\begin{aligned} \frac{d}{dx}\left(y(x)\right) &= \frac{d}{dx}\left((x^3 + 2x)^{10}\right) \\ &= \frac{d}{dx}\left(f(g(x))\right) \\ &= f'(g(x))g'(x) \\ &= 10(x^3 + 2x)^9 \cdot (3x^2 + 2). \end{aligned}$$

Essentially, what we have done is to substitute $g(x) = x^3 + 2x$ in our function for $y(x)$, to make the differentiation easier.

Example
To find

$$\frac{d}{dx}\left(\sin(1 + x^2)\right)$$

take $f(w) = \sin w$ and $g(x) = 1 + x^2$ ($f'(w) = \cos w$ and $g'(x) = 2x$).
Then

$$\begin{aligned} \frac{d}{dx}\left(\sin(1 + x^2)\right) &= \frac{d}{dx}\left(f(g(x)\right) \\ &= f'(g(x))g'(x) \\ &= \cos(1 + x^2) \cdot 2x \\ &= 2x\cos(1 + x^2) \end{aligned}$$