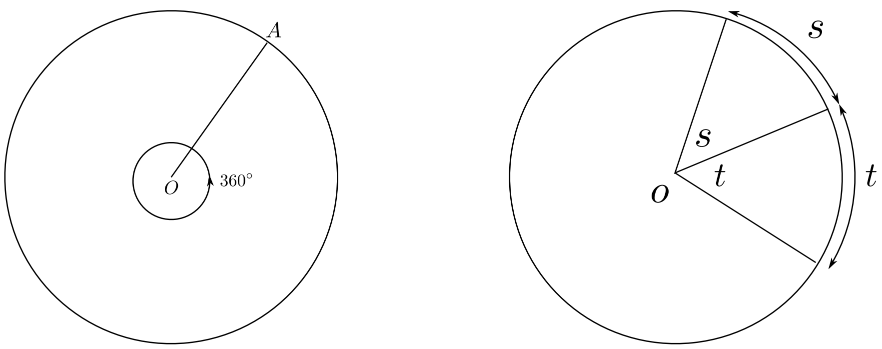
1.6 Trigonometric functions

1.6.1 Measuring angles

Definition: Degrees
Degrees are defined so that one full turn is $360°$.

Definition: Radians
Radians (usually abbreviated as rad or ^{c}) are defined using a circle of radius 1. The angle between two radii in the unit circle in radians is equal to the arc length between the two radii.



(a) 360° is a full turn.

(b) A unit circle with angles of s^c and t^c marked.

We can see immediately that a full turn is 2π rad because a circle of radius 1 has a circumference 2π. Therefore we have

$$\begin{aligned} 1 \text{ turn} &= 360° = 2\pi \,\text{rad} \\ \tfrac{1}{2} \text{ turn} &= 180° = \pi \,\text{rad} \end{aligned}$$

So,

$$\begin{aligned} 1\,\text{rad} &= \frac{180°}{\pi} \\ 1° &= \frac{\pi}{180}\,\text{rad} \end{aligned}$$

If the radius is not 1, then you need to take the ratio

$$\frac{\text{arc length}}{\text{radius}} = \text{angle (in radians)}. \tag{1.11}$$

1.6.2 Trignometric functions: cosine, sine & tangent

Definition: sin, cos and tan
The values $\cos(\theta)$ and $\sin(\theta)$ (often written $\cos\theta$, $\sin\theta$) are the horizontal and vertical coordinates of the point C. $\tan(\theta)$ (often $\tan\theta$) is defined to be $\frac{\sin\theta}{\cos\theta}$.

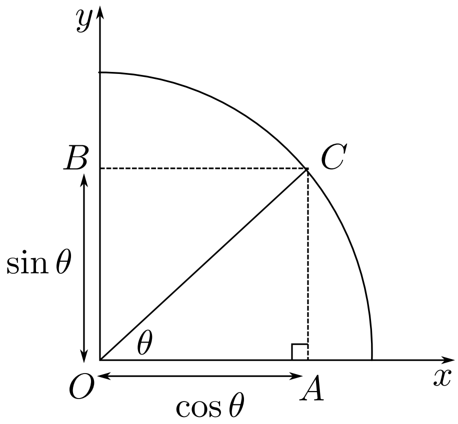


Figure 1.6: Geometric definition of cos and sin, circle radius $r = OC = 1$.

It can easily be seen that this definition is equivalent to the "SOH CAH TOA" definition you are familiar with:

$$\sin\theta = \frac{AC}{OC} = AC$$
$$\cos\theta = \frac{OA}{OC} = OA$$
$$\tan\theta = \frac{AC}{OA} = \frac{\sin\theta}{\cos\theta}$$

Although this definition allows for sin, cos and tan to easily be extended to angles outside the range $\left[0, \frac{\pi}{2}\right]$

1.6.3 Properties of sin, cos and tan

Property
$\cos^2\theta + \sin^2\theta = 1$

Proof: Use Pythagoras' Theorem in triangle OAC. □

Property
cos and sin are periodic functions with period 2π (i.e. for any x, $\cos(x+2\pi) = \cos x$, $\sin(x+2\pi) = \sin x$).

Property
$\cos : \mathbb{R} \to [-1, 1]$ and $\sin : \mathbb{R} \to [-1, 1]$.

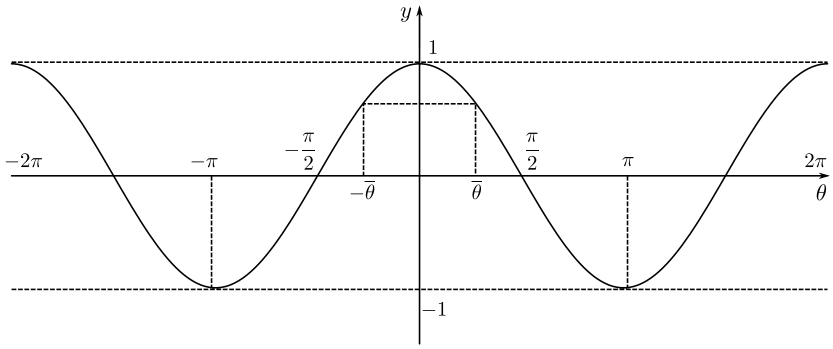


Figure 1.7: Graph of $\cos\theta$.

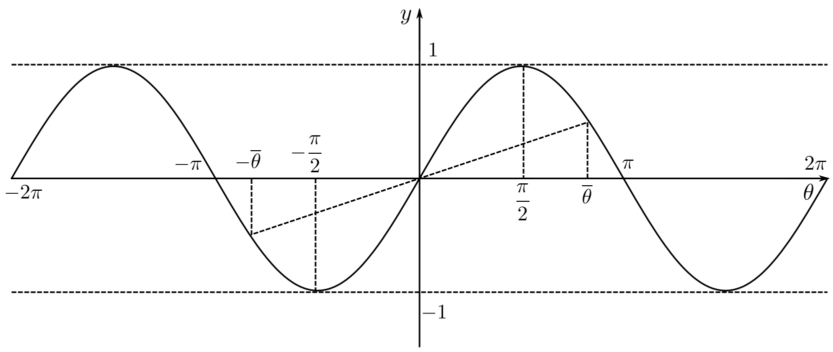


Figure 1.8: Graph of $\sin\theta$.

Property
cos is an even function. sin is an odd function.

Property
cos and sin are the same shape but shifted by $\pi/2$, which means

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$$

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

Property: Addition formulae

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

Property: Double angle formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$

Proof:

$$\begin{aligned} \cos 2\theta &= \cos(\theta + \theta) \\ &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$

$$\begin{aligned} \sin 2\theta &= \sin(\theta + \theta) \\ &= 2\sin\theta\cos\theta \end{aligned}$$

□

Property: Half angle formulae

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos\alpha}{2}$$
$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{2}$$

Proof: Let $2\theta = \alpha$, then

$$\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 \quad \Longrightarrow \quad \cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos\alpha}{2}$$
$$\cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \quad \Longrightarrow \quad \sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{2}$$

□

Property
tan has vertical asymptotes at $\theta = \frac{\pi}{2}(2N - 1)$ for $N \in \mathbb{Z}$.

Proof: At $\theta = \frac{\pi}{2}(2N-1)$, $\cos\theta = 0$.

□

Property
$\tan : \mathbb{R} \setminus \{\frac{\pi}{2}(2N-1) : N \in \mathbb{Z}\} \to \mathbb{R}$

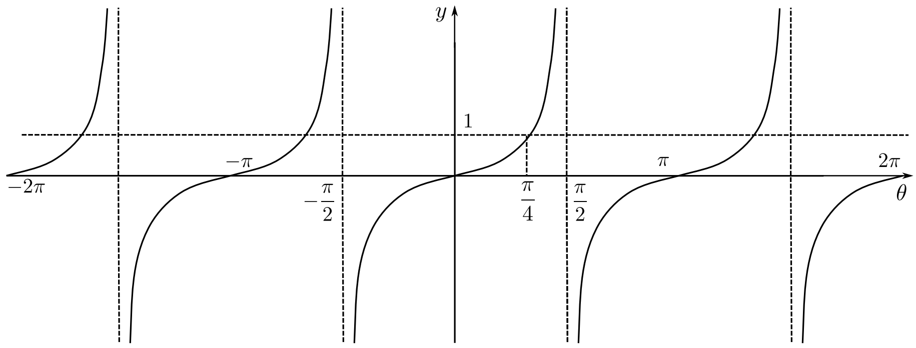


Figure 1.9: Graph of $\tan\theta$.

Property
tan is periodic with period π.

Property: Double angle formula for tan

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

Proof:

$$\begin{aligned} \tan(\theta+\phi) &= \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} \\ &= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}. \end{aligned}$$

Now divide by $\cos\theta\cos\phi$.

□

Definition
Secant, cosecant and cotangent The secant, cosecant and cotangent functions are defined as

$$\sec x = \frac{1}{\cos x}, \quad \text{cosec} x = \frac{1}{\sin x}, \quad \text{cot} x = \frac{1}{\tan x}.$$

Property

$$1 + \tan^2 x = \sec^2 x.$$

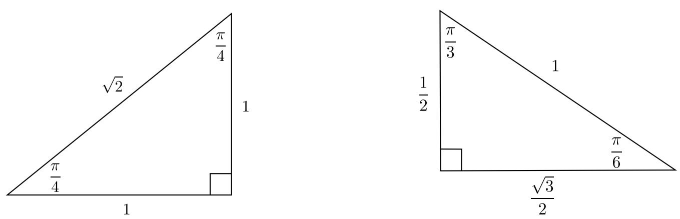
Proof: Divide $\cos^2\theta + \sin^2\theta = 1$ through by $\cos^2 x$. $\square$

Property

There are a number of “special angles” for which you should remember the values of sin, cos and tan:

Angle (°)	Angle (c)	sin	cos	tan
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{3}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	∞

These can be easily remembered via the following triangles:



(a) $\pi/4$ triangle.

(b) $\pi/3$, $\pi/6$ triangle.

Figure 1.10: Some well known results for particular angles can be derived by the above triangles for sin, cos and tan.