4.5 Solving initial-value problems numerically

Most differential equations can not be solved analytically, so we try to solve them numerically.

4.5.1 Euler's method

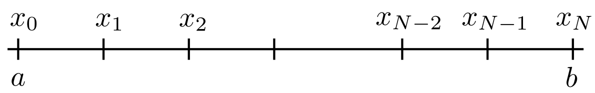
Suppose we have an initial-value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = y_0.$$

We want to find the solution $y(x)$ numerically on the interval $[a, b]$.

First we divide $[a, b]$ into N equal subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_k < \cdots < x_N = b.$$



Similarly to when we looked at the trapezium method:

$$x_k = a + kh, \quad h = \frac{b-a}{N} \quad \text{(step size)}, \quad k = 0, 1, \ldots, N.$$

If h is small, then the curve will be close to a straight line between x_k and x_{k+1}. The differential equation tells us that the gradient at x_k is $f(x_k, y_k)$. Therefore we approximate the curve between x_k and x_{k+1} by a straight line with gradient $f(x_k, y_k)$.

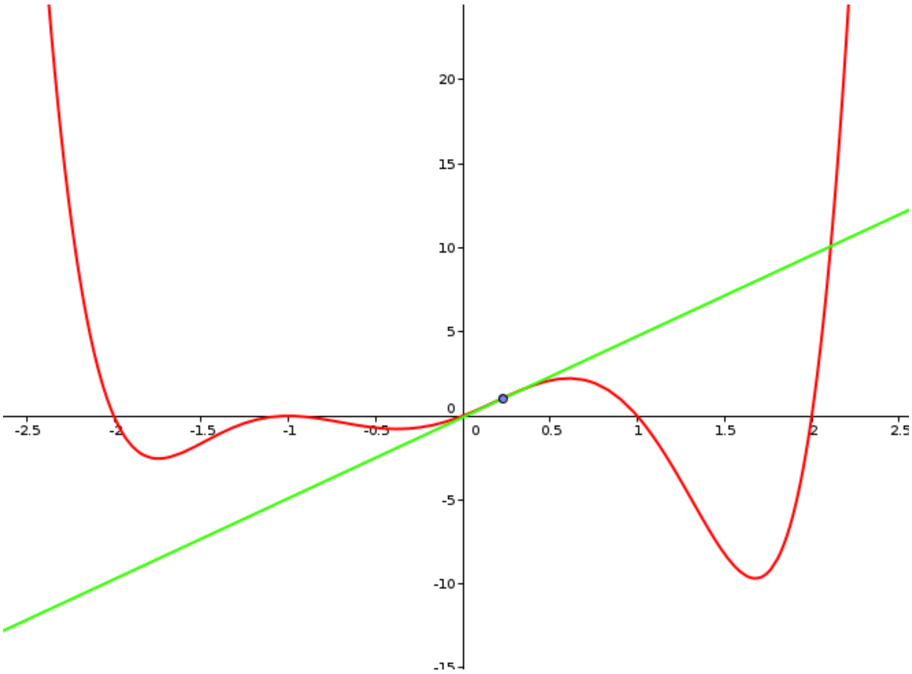


Figure 4.2: Near the point marked, the curve (red) is closely approximated by the tangent (green).

This gives us the following approximation for y_{k+1}:

$$y_{k+1} = y_k + hf(x_k, y_k)$$

We know that $y_0 = a$. We can then use this formula to approximate y_1, then use it again to find y_2, then y_3 and so on. The method of approximating y_k iteratively in this way is called **Euler's method**.

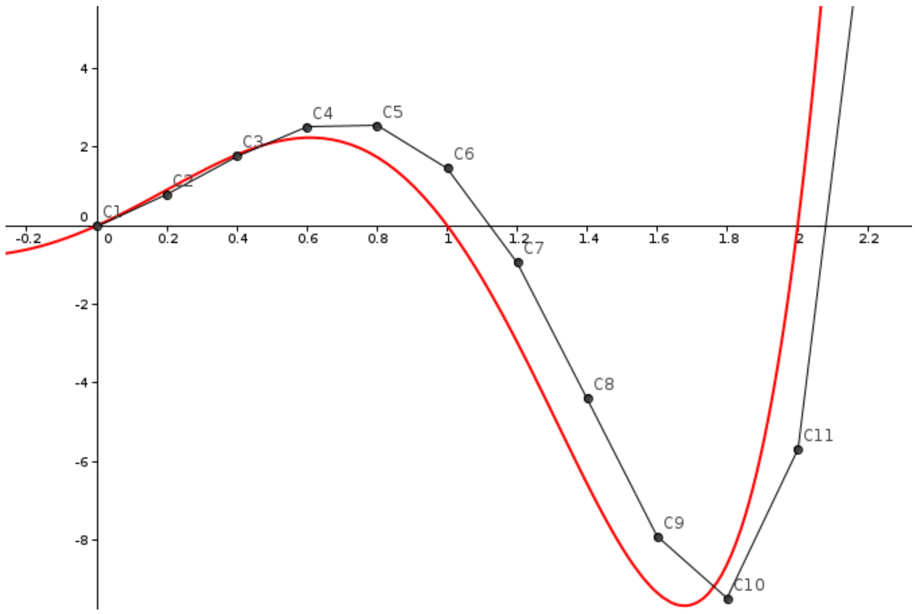


Figure 4.3: Using Euler's method to approximate a curve, starting at 0. at each point, a straight line is drawn with the gradient equal to that of the curve.

Example
Estimate $y(1)$, where $y(x)$ satisfies the initial-value problem:

$$\frac{dy}{dx} = y, \quad y(0) = 1.$$

We know the exact solution is

$$y(x) = e^x, \quad \Longrightarrow \quad y(1) = e \approx 2.71828.$$

Now we apply Euler's method to the problem. We have

$$f(x, y) = y.$$

First, we take $N = 5$, then $h = (1 - 0)/5 = 0.2$.

x_0	x_1	x_2	x_3	x_4	x_5
0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1

$$\begin{aligned}
y_0 &= y(0) = 1 \\
y_1 &= y_0 + hf(x_0, y_0) = 1 + 0.2 \times 1 = 1.2 \\
y_2 &= y_1 + hf(x_1, y_1) = 1.2 + 0.2 \times 1.2 = (1.2)^2 \\
y_3 &= y_2 + hf(x_2, y_2) = y_2 + hy_2 = y_2(1 + h) = (1.2)^2 \times 1.2 = (1.2)^3 \\
y_4 &= (1.2)^4 \\
y_5 &= (1.2)^5 \approx 2.48832.
\end{aligned}$$

Euler's method with 5 subintervals has given us the approximation

$$y(1) \approx 2.48832.$$

As we know the exact solution, we can look at the error:

$$\begin{aligned} \text{error} &= e - y_5 \\ &= 2.71828 - 2.48832 \\ &= 0.22996 \end{aligned}$$

Now, we double the number of subintervals: $N = 10$, $h = 0.1$ then we need 10 steps to reach $x_{10} = 1$.

$$y_{10} = (1.1)^{10} \approx 2.59374,$$

then we have

$$\text{error} = 2.71828 - 2.59374 = 0.12454.$$

For $N = 20$, $h = 0.05$ and so

$$y_{20} = (1.05)^{20} \approx 2.65330, \quad \text{error} = 0.0650.$$

For $N = 40$, $h = 0.025$ and so

$$y_{20} = (1.025)^{40} \approx 2.68506, \quad \text{error} = 0.0332.$$

As we increase the number of intervals, the value becomes a better approximation.