4.3 Complementary functions and particular integrals

In the previous three examples, we gained the following results:

$$y = \frac{1}{3}x^2 + c \cdot .\frac{1}{x},$$

$$y = 1 + c \cdot e^{-\frac{1}{2}x^2},$$

$$y = -(x^2 + 2x + 2) + ce^x.$$

These examples have something very important in common, that is the solutions have the following form

$$y = f(x) + cg(x),$$

with explicit functions f and g.

Definition
When $y = f(x) + cg(x)$ is the solution of an ODE, f is called the **particular integral** (P.I.) and g is called the **complementary function** (C.F.).

We can use particular integrals and complementary functions to help solve ODEs if we notice that:

1. The complementary function (g) is the solution of the homogenous ODE.
2. The particular integral (f) is any solution of the non-homogenous ODE.

Example
We will use complementary functions and particular integrals to solve

$$y' + \lambda y = p(x), \quad \lambda \text{ is constant.}$$

We know that the general solution is

$$y(x) = \underbrace{f(x)}_{\text{particular integral (P.I.)}} + \underbrace{Cg(x)}_{\text{complementary function (C.F.)}},$$

where

$$f' + \lambda f = p(x),$$

$$g' + \lambda g = 0,$$

and C is the constant of integration.
We start by finding g. We need to solve

$$g' + \lambda g = 0.$$

The solution of this is

$$g = Ce^{-\lambda x}.$$

[This can be found be separating the variables or by inspection.]

Therefore, the general solition to $y' + \lambda y = p(x)$ is

$$y = f(x) + Ce^{-\lambda x},$$

where f is a particular integral.

The value of f depends on p. To find f, we make a guess at what it might look like then see if we are right.

Example: $p(x) = x$
Consider the differential equation

$$y' + y = x,$$

so we have

$$\lambda = 1, \quad p(x) = x.$$

The solution of this ODE will be of the form

$$y = f(x) + Ce^{-x}.$$

We can guess that f should be a polynomial with a degree of one, since $p(x) = x$. So we try the most general first order polynomial, $f(x) = ax + b$, and so $f'(x) = a$. Substituting $y = f(x)$ into the differential equation we have that

$$\begin{aligned} f' + f &= a + ax + b \\ &= ax + (a + b). \end{aligned}$$

And so

$$x \equiv ax + (a + b).$$

Thus, comparing coefficients from the LHS and RHS we must have that

$$\left.\begin{array}{l} a = 1 \\ a + b = 0 \end{array}\right\} \quad \Longrightarrow \quad \left.\begin{array}{l} a = 1 \\ b = -1 \end{array}\right\} \quad \Longrightarrow \quad f(x) = x - 1,$$

so $f(x) = x - 1$ is the particular integral. Therefore, the general solution to the original equation is

$$y(x) = x - 1 + Ce^{-x}.$$

When we solve higher order linear ODEs, we use a similar method:]

1. We solve the homogenous equation to find the complementary function.
2. We guess the form of the particular integral then try it.