Chapter 4

Differential Equations

In many applications, we have equations relating a functions and its derivatives. For example:

> In radioactive decay, we know $\frac{dy}{dt} = \lambda y$, where y is the number of particles of radioactive material.
>
> Inflation is expressed as a percentage of current prices, so $\frac{dp}{dt} = ip$, where p is prices and i is inflation.
>
> The movement of an object on a spring follows the equation $\frac{d^2y}{dx^2} = -\omega y$.

Equations like these are called **(ordinary) differential equations** or **ODEs**.

In this chapter we will look at methods for solving ODEs.

4.1 Terminology

Definition
An equation involving y and $\frac{dy}{dx}$ is called a **first order** ODE.
An equation involving y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ is called a **second order** ODE.
When solving ODEs, solutions involving constants often appear. These are called **general solutions** of ODEs.

Extra information is often given to give the constants in the general solution a value.

Definition
The extra information given is called the **boundary conditions**.
A problem with an ODE and boundary conditions is called an **initial value prob-**

lem or **IVP**.

Definition
An n-th order differential equation is linear if it can be written in the form:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_{n-2}(x)y^{(n-2)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

or

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x),$$

where a_i $(i = 0, 1, 2, \ldots, n)$ and f are known functions of x.

Example
$y' + 2y = e^x$ is a first-order linear differential equation.
$yy' = x$ is a first-order non-linear differential equation.
$y'' - e^x y' + y = x$ is a second-order linear differential equation.

Definition
If $f(x) \equiv 0$, then the differential equation is said to be **homogeneous**; otherwise, we say the equation is **non-homogenous** or **inhomogenous**.

Example
$y' + 2y = e^x$ is a non-homogeneous differential equation.
$y' + 2y = 0$ is a homogeneous differential equation.

4.2 First order differential equations

Here we will consider different techniques to solve first order ODEs.

4.2.1 Separation of variables

First order ODEs can be written in the form

$$\frac{dy}{dx} = f(x, y).$$

For example

$$y' = -2xy + e^x,$$
$$\frac{dy}{dx} = \pm\sqrt{x^3 - 2\ln y + 4e^x},$$
$$y' = x/y^2.$$

Definition
A function $f(x,y)$ is **separable** if it can be written as

$$f(x,y) = g(x)h(y).$$

Example
Let $f(x,y) = \dfrac{x}{y^2} = x \cdot \dfrac{1}{y^2}$.
This is seperable because

$$\frac{x}{y^2} = x \cdot \frac{1}{y^2}$$

so $f(x,y) = g(x)h(y)$ where

$$g(x) = x$$
$$h(y) = \frac{1}{y^2}$$

When f is seperable, we can solve $\dfrac{dy}{dx} = f(x,y)$ by a method called **separating the variables**.

Example
Consider the differential equation

$$y' = \lambda y.$$

We already know the solution to this equation. Now let us see how to derive it using separation of variables.

$$\frac{dy}{dt} = \lambda y,$$

taking all things relating to y to the left, and for t to the right, we have

$$\frac{1}{y} dy = \lambda dt.$$

Integrating both sides we have

$$\int \frac{1}{y}\ dy = \int \lambda\ dt,$$

hence, using what we have learnt in previous chapters we get

$$\ln y = \lambda t + C.$$

Finally, re-arranging for y, we have

$$y = e^{\lambda t + C} = Ae^{\lambda t}, \quad A = e^C.$$

Example
Consider the equation
$$\frac{dy}{dx} = xy,$$
following the procedure as in the previous example, we have
$$\frac{1}{y}\ dy = x\ dx.$$
Integrating both sides we have
$$\begin{aligned} \int \frac{1}{y}\ dy &= \int x\ dx, \\ \implies \qquad \ln y &= \frac{1}{2}x^2 + C. \end{aligned}$$
Taking exponentials of both sides in order to re-arrange for y, we get
$$y = e^{\frac{1}{2}x^2+C} = Ae^{\frac{1}{2}x^2}, \quad A = e^C.$$
We can check if this satisfies the original equation:
$$\frac{dy}{dx} = \frac{d}{dx}\left(Ae^{\frac{1}{2}x^2}\right) = Ae^{\frac{1}{2}x^2+} \cdot \frac{1}{2} \cdot 2x = xy.$$

Example
Consider the differential equation
$$y^2y' = x.$$
We first write it in the form $y' = f(x, y)$, i.e.
$$\frac{dy}{dx} = \frac{x}{y^2},$$
now we realises that we can apply separation of variable, so
$$y^2dy = x\ dx,$$
$$\begin{aligned} \implies \qquad \int y^2dy &= \int x\ dx \\ \implies \qquad \frac{1}{3}y^3 &= \frac{1}{2}x^2 + C, \\ \implies \qquad y &= \left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}, \end{aligned}$$

where C' is some constant (different to C, since we multiplied through by 3). Again, we check the solution satisfies the equation

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right] \\
&= \frac{1}{3}\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \cdot \frac{3}{2} \cdot 2x \\
&= x\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \\
&= x\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right]^{-2} \\
&= \frac{x}{y^2}.
\end{aligned}$$

Example

Consider the following initial-value problem:

$$\frac{dy}{dx} = y^2(1 + x^2), \quad y(0) = 1.$$

First, we find the general solution, note, we can use separation of variables in this example, so

$$\frac{1}{y^2}\ dy = (1 + x^2)\ dx$$

$$\begin{aligned}
\implies \quad & \int \frac{1}{y^2}\ dy = \int (1 + x^2)\ dx \\
\implies \quad & -\frac{1}{y} = x + \frac{1}{3}x^3 + C \\
\implies \quad & y = -\frac{1}{x + \frac{1}{3}x^3 + C}.
\end{aligned}$$

Now check that the general solution satisfies the original differential equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{1}{x + \frac{1}{3}x^3 + C}\right) = \frac{1 + x^2}{\left(x + \frac{1}{3}x^3 + C\right)^2} = (1 + x^2)y^2.$$

Now it remains to find the constant C, by applying the condition $y(0) = 1$, i.e. we put $x = 0$.

$$y(0) = -\frac{1}{0 + \frac{1}{3} \cdot 0^3 + C} = -\frac{1}{C} = 1, \quad \implies \quad C = -1.$$

So the solution to the initial value problem is

$$y = \frac{1}{1 - x - \frac{1}{3}x^3}.$$

Example
Consider the initial value problem

$$e^y y' = 3x^2, \quad y(0) = 2.$$

First, find the general solution,

$$\begin{aligned} & e^y y' = 3x^2 \\ \implies \quad & \frac{dy}{dx} = 3x^2 e^{-y} \\ \implies \quad & e^y \; dy = 3x^2 \; dx \\ \implies \quad & \int e^y \; dy = \int 3x^2 \; dx \\ \implies \quad & e^y = x^3 + C \\ \implies \quad & y = \ln(x^3 + C). \end{aligned}$$

Check:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(x^3 + C)\right) = \frac{3x^2}{x^3 + C} = 3x^2 \frac{1}{x^3 + C}.$$

Recall $e^{\ln(a)} = a$, using this, we can write

$$\frac{dy}{dx} = 3x^2 e^{\ln\left(\frac{1}{x^3+C}\right)} = 3x^2 e^{-\ln(x^3+C)} = 3x^2 e^{-y}.$$

Now we apply the initial condition,

$$y(0) = \ln(C) = 2 \quad \implies \quad C = e^2,$$

so we have the final solution

$$y(x) = \ln(x^3 + e^2).$$