MATH6103 Differential & Integral Calculus
MATH6500 Elementary Mathematics for Engineers

Department of Mathematics,
University College London

Matthew Scroggs
web: www.mscroggs.co.uk/6103
e-mail: matthew.scroggs.14@ucl.ac.uk

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These notes are based on the original notes by Ali Khalid which can be found at www.homepages.ucl.ac.uk/akhalid.

Contents

Chapter 1

Functions

1.1 What is a function?

A function takes an input, does something to it, then gives an output.

Example
If the function "add two" is given 4 as an input it would give 6 as its output.
We will represent this function as $f(x) = x + 2$.

1.2 Domains, ranges and variables

In this section, we will look at some properties of functions. First we define some notation we will use when writing sets.

Definition
A **set** is a collection of objects, often numbers. An object in a set is called an element of the set.
Sets are written as a list of items inside curly braces: { and }.
If an object x is in a set A, we will write $x \in A$. If an object y is not in a set A, we will write $x \notin A$.

Example
The following are examples of sets:

(i) $A = \{1, 2, 8\}$ is a set containing the numbers 1, 2 and 8. We could write $8 \in A$ and $4 \notin A$;

(ii) $\mathbb{Z} = \{\text{all whole numbers}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$;

(iii) $\mathbb{N} = \{\text{all positive whole numbers}\} = \{0, 1, 2, 3, \dots\}$;

(iv) $\mathbb{R} = \{\text{all real numbers}\}$.

Definition
If we have two sets A and B, then a **function** $f : A \to B$ is a rule that sends each element x in A to exactly one element called $f(x)$ in B.
We call the set A the **domain** of f. The **range** of f is the set of all values which f gives as an output (B or a subset of B).
If $x \in A$ then x is called an **independent variable**. If y is in the range of f then y is called a **dependent variable**.

1.3 Representing a function

Functions are sometimes represented by mapping diagrams:

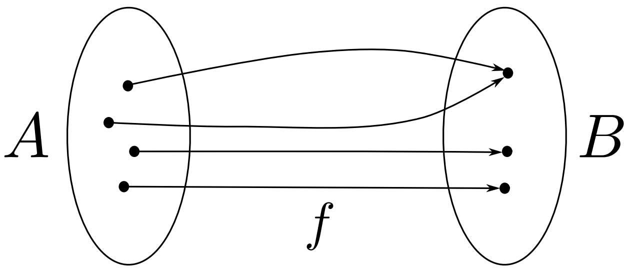


Figure 1.1: *Function f 'maps' elements from set A on to elements in set B. Many-to-one relationship.*

The mapping diagram shows the domain and range of the function. The arrows show where is member of A is sent by the function.

Definition
Let $f : A \to B$ be a function.
If, whenever x and y are in A and not equal, $f(x)$ and $f(y)$ are not equal, the function is called **one-to-one**.
If there are an x and a y in A which are not equal, but $f(x)$ and $f(y)$ are equal, the function is called **many-to-one**.

Definition
Let $f : A \to B$ be a function.
If $f(-x) = f(x)$, f is called an **even** function.
If $f(-x) = -f(x)$, f is called an **odd** function.

Definition
If $f(x + T) = f(x)$ for all x, then we say that $f(x)$ is a **periodic function** with

period T.

Definition
A function, f, has a **vertical asymptote** at $x = a$ if as $x \to a$, $f(x) \to \pm\infty$.
A function, f, has a **horizontal asymptote** at $y = b$ if as $x \to \infty$, $f(x) \to b$ or if as $x \to -\infty$, $f(x) \to b$.

1.4 Polynomials

Definition
A polynomial is a function P with a general form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{1.1}$$

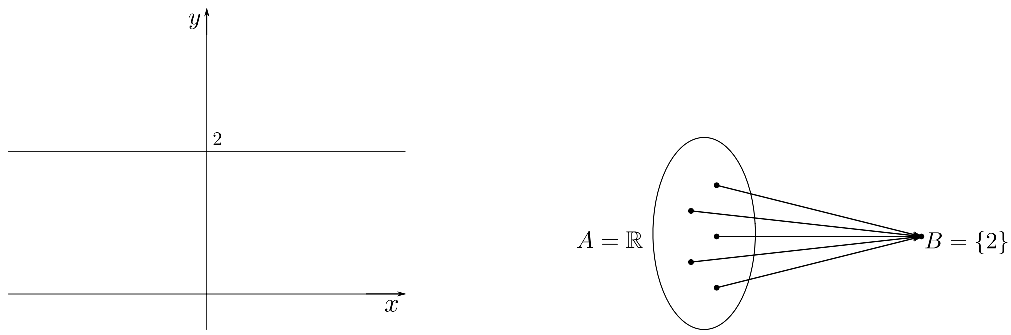
where the coefficients a_i $(i = 0, 1, \ldots, n)$ are numbers and n is a non-negative whole number. The highest power whose coefficient is not zero is called the **degree** of the polynomial.

Example

$P(x)$	2	$3x^2 + 4x + 2$	$\frac{1}{1+x}$	$\sqrt{x}$	$1 - 3x + \pi x^3$	$2t + 4$
Polynomial?	Yes	Yes	No	No	Yes	Yes
Order	0	2	N/A	N/A	3	1

1.4.1 Degree 0 polynomials

$P_0(x) = a_0 x^0 = a_0$, say $P(x) = 2$. This polynomial is simply a constant. Degree 0 polynomials are not very interesting.



(a) $y = P(x) = 2$. (b) *Entire domain mapped to one point.*

Figure 1.2: *A degree 0 polynomial.*

1.4.2 Degree 1 polynomials

$P_1(x) = a_0x^0 + a_1x^1 = ax + b$ ($a \neq 0$). These are called *linear*, since the graph of $y = ax + b$ is a straight line. The linear equation $ax + b = 0$ has solution $x = -b/a$.

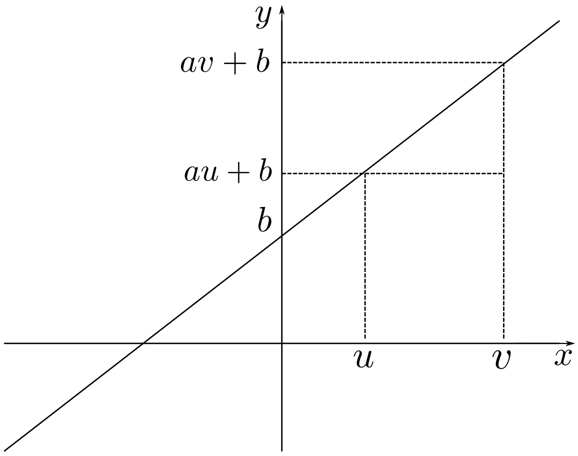


Figure 1.3: *Linear graph given by* $y = ax + b$.

The **gradient** of $y = ax + b$ is a. It can be worked out as follows:

$$\begin{aligned} \text{gradient} &= \frac{\text{change in height}}{\text{change in distance}} \\ &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{(av+b)-(au+b)}{v-u} \\ &= \frac{a(v-u)}{v-u} = a. \end{aligned} \tag{1.2}$$

1.4.3 Degree 2 polynomials

$P_2(x) = a_0x^0 + a_1x^1 + a_2x^2 = ax^2 + bx + c$, $a \neq 0$. This is known as a quadratic polynomial. There are three common ways to solve the quadratic equation $ax^2 + bx + c = 0$:

1. Factorising

Some (but not all) quadratics can be factorised.

To factorise $x^2 + bx + c$, look for r_1 and r_2 such that:

- $r_1 r_2 = c$
- $r_1 + r_2 = b$

Then $x^2 + bx + c = (x + r_1)(x + r_2)$

Proof: Expanding the brackets in $(x + r_1)(x + r_2)$ gives:

$$x^2 + (r_1 + r_2)x + r_1 r_2.$$

Setting this equal to $x^2 + bx + c$ gives the conditions above.

□

Example
Let $P(x) = x^2 - 3x + 2$.
P can be factorised as $P(x) = (x - 2)(x - 1)$.
The solutions of $P(x) = 0$ are $x = 2$ and $x = 1$.
note: In this example, b is negative.

2. Completing the square

Completing the square is best demonstrated with an example:

Example
Let $P(x) = x^2 - 3x + 2$ ($b = -3$,$c = 2$).
First, add and subtract $\left(\frac{b}{2}\right)^2$ from $P(x)$:

$$P(x) = x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 + 2$$

This has not changed the value of P as we have added and subtracted the same thing, but it allows us to factorise the first three terms, giving:

$$P(x) = \left(x - \frac{3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 + 2$$

We have completed the square. If we now need to solve $P(x) = 0$:

$$\left(x - \frac{3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 + 2 = 0 \tag{1.3}$$

$$\left(x - \frac{3}{2}\right)^2 = \left(\frac{-3}{2}\right)^2 - 2 \tag{1.4}$$

$$= \frac{1}{4} \tag{1.5}$$

$$x - \frac{3}{2} = \pm\frac{1}{2} \tag{1.6}$$

$$x = \frac{3}{2} \pm \frac{1}{2} \tag{1.7}$$

$$x = 1 \text{ or } 2 \tag{1.8}$$

3. The quadratic formula

Completing the square then with a general quadratic equation $ax^2 + bx + c = 0$ gives the quadratic formula for the solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.9}$$

Sketching a quadratic

The graph of $P(x) = x^2 - 3x + 2$ looks like:

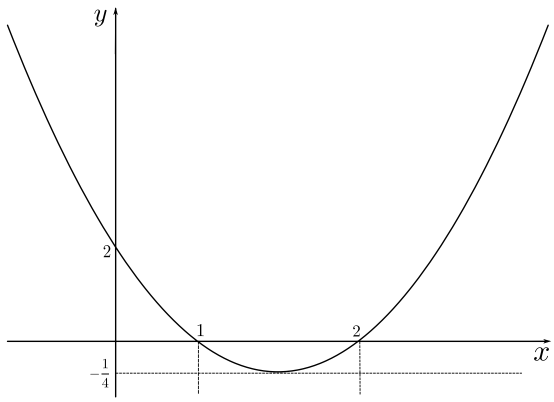


Figure 1.4: *Quadratic graph given by* $y = x^2 - 3x + 2$.

We know (from above):

(i) the graph is a "cup" rather than a "cap" since the coefficient of x^2 is positive. Also we can easily see $P(0) = 2$;

(ii) $P(x) = 0$ at $x = 1$ and $x = 2$;

(iii) $P(x)$ is minimal at $x = \frac{3}{2}$ and $P\left(\frac{3}{2}\right) = -\frac{1}{4}$. Note,

$$\left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4},$$

since anything squared is always positive!

1.4.4 Complex numbers

To allow ourselves to solve all quadratics, not just those with real roots, we introduce i:

Definition: i

$$\mathrm{i} = \sqrt{-1}$$

Numbers of the form $b\mathrm{i}$ are called imaginary numbers. Numbers of the form $a + b\mathrm{i}$ are called complex numbers. They can be used as follows:

Example
To solve $x^2 + 4 = 0$:

$$\begin{aligned}x^2 + 4 &= 0\\ x^2 &= -4\\ x &= \pm\sqrt{-4}\\ x &= 2\text{i or } -2\text{i}\end{aligned}$$

To solve $x^2 + 6x + 13 = 0$:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-6 \pm \sqrt{36 - 4 \times 13}}{2}\\ &= \frac{-6 \pm \sqrt{-16}}{2}\\ &= \frac{-6 \pm 4\text{i}}{2}\\ &= -3 + 2\text{i or } -3 - 2\text{i}\end{aligned}$$

To add, multiply and subtract complex numbers, treat i like a variable and remember that $\text{i}^2 = -1$:

Example

$$(2 + 4\text{i}) + (3 - 2\text{i}) = 5 + 2\text{i}$$

$$\begin{aligned}(2 + 4\text{i}) \times (3 - 2\text{i}) &= 6 + 12\text{i} - 4\text{i} - 8\text{i}^2\\ &= 6 + 12\text{i} - 4\text{i} + 8\\ &= 14 + 8\text{i}\end{aligned}$$

1.4.5 Degree ≥ 3 polynomials

In general, we have the algebraic equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0, \tag{1.10}$$

which has n roots, including real and complex roots.

$n = 2$	we have formulae for roots	(quadratics)
$n = 3$	we have formulae for roots	(cubic)
$n = 4$	we have formulae for roots	(quartics)
$n > 4$	No general formulae exist	(proven by Évariste Galois)

But in any case, we may try factorisation to find the roots. We have the useful theorem:

Theorem: Factor theorem
Let P be a polynomial of degree n. For $a \in \mathbb{R}$ (or C),

$$P(a) = 0 \quad \text{if and only if} \quad P(x) = (x-a)Q(x)$$

where Q is a polynomial of degree $n-1$.

Once one root is found, this theorem can be used to factorise the polynomial.

Example
Consider $P(x) = x^3 - 8x^2 + 19x - 12$. We know that $x = 1$ is a solution to $P(x) = 0$, then it can be shown that

$$P(x) = (x-1)Q(x) = (x-1)(x^2 - 7x + 12).$$

Here $P(x)$ is a cubic and thus $Q(x)$ is a quadratic.

The next examples show two methods of finding $Q(x)$.

Example: Comparing coefficients
Consider $P(x) = x^3 - x^2 - 3x - 1$. By observation, we know

$$P(-1) = (-1)^3 - (-1)^2 - 3(-1) - 1 = 0.$$

So $x_1 = -1$ is a root. Let us write

$$P(x) = (x+1)(ax^2 + bx + c),$$

then multiplying the brackets we have

$$P(x) = ax^3 + (a+b)x^2 + (b+c)x + c,$$

which should be equivalent to $x^3 - x^2 - 3x - 1$. Thus, comparing the corresponding coefficients we have

$$\begin{aligned} a &= 1, \\ a+b &= -1, \\ b+c &= -3, \\ c &= -1. \end{aligned}$$

This set of simultaneous equations has the solution

$$a = 1, \quad b = -2, \quad c = -1.$$

So we can write

$$P(x) = (x+1)(x^2 - 2x - 1)$$

To find the other two solutions of $P(x) = 0$, we must set $(x^2 - 2x - 1) = 0$ which has solutions $x_{2,3} = 1 \pm \sqrt{2}$, together with $x_1 = -1$ we have a complete set of solutions for $P(x) = 0$.

As with many areas of mathematics, there are many ways to tackle a problem. Another way to find $q(x)$ given you know some factor of $P(x)$, is called *polynomial division*.

Example: Long division of polynomials
Consider $P(x) = x^3 - x^2 - 3x - 1$, we know $P(-1) = 0$. The idea is that we "divide" $P(x)$ by the factor $(x + 1)$, like so:

$$
\begin{array}{r}
x^2 - 2x - 1 \\
x + 1 \overline{)\, x^3 - x^2 - 3x - 1} \\
-\, x^3 - x^2 \\
\hline
-\, 2x^2 - 3x \\
2x^2 + 2x \\
\hline
-\, x - 1 \\
x + 1 \\
\hline
0
\end{array}
$$

Hence, multiplying the *quotient* by the *divisor* we have $(x + 1)(x^2 - 2x - 1) = x^3 - x^2 - 3x - 1$.

1.5 Exponentials

We have seen functions of the form $f(x) = x^a$, where a is a constant. What happens if we swap the a and the x and look at $f(x) = a^x$?

First, let's look at what a^x means for all values of $x \in \mathbb{R}$.

1.5.1 Indices

when we wish to multiply a number by itself several times, we make use of index or power notation. We have notation for powers (for $a \in \mathbb{R}$:

$$
\begin{aligned}
a^2 &= a \cdot a \\
a^3 &= a \cdot a \cdot a \\
a^x &= \overbrace{a \cdot a \ldots a \cdot a}^{x} \qquad x \in \mathbb{N} \text{ and } x \neq 0
\end{aligned}
$$

Here, a is called the **base** and x is called the **index** or **power**. We also know the following properties:

Properties of exponents
For any $a \in \mathbb{R}$ and $x, y \in \mathbb{R}$

1. $a^{x+y} = a^x \cdot a^y$

2. $(a^x)^y = a^{xy}$
3. $a^x \cdot b^x = (ab)^x$

Examples

1. $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^3 \cdot 2^2$.
2. $3^6 = 3^{2\times 3} = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^2 \cdot 3^2 \cdot 3^2 = (3^2)^3$.
3. $2^3 \cdot 3^3 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = (2 \cdot 3)^3$.

We can use the properties of exponents to justify the definitions of a^x when x is not a positive integer. Throughout this we will assume that $a > 0$.

For $x \in \mathbb{Z}$

For $x = 0$, we notice that:

$$\begin{aligned} a^2 &= a^{2+0} \\ &= a^2 \cdot a^0 \end{aligned}$$

This shows that:

$$a^0 = 1$$

When x is a negative integer:

$$\begin{aligned} 1 &= a^0 \\ &= a^{x-x} \\ &= a^x \cdot a^{-x} \end{aligned}$$

Dividing by a^x, we get:

$$a^{-x} = \frac{1}{a^x}$$

For $x \in \mathbb{Q}$ (the set of all fractions)

For $x = \frac{1}{n}$, notice that:

$$\begin{aligned} \left(a^{\frac{1}{n}}\right)^n &= a^{\frac{1}{n}\cdot n} \\ &= a^1 \\ &= a \end{aligned}$$

Taking the nth root gives:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Similarly, we find that

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

For $x \in \mathbb{R}$

If x is an irrational number, then, for any small number $\epsilon > 0$ we can always find two rational numbers c and d which satisfy $x - \epsilon < c < x < d < x + \epsilon$. a^x is defined to be the limit of a^c (or a^d) as $\epsilon \to 0$.

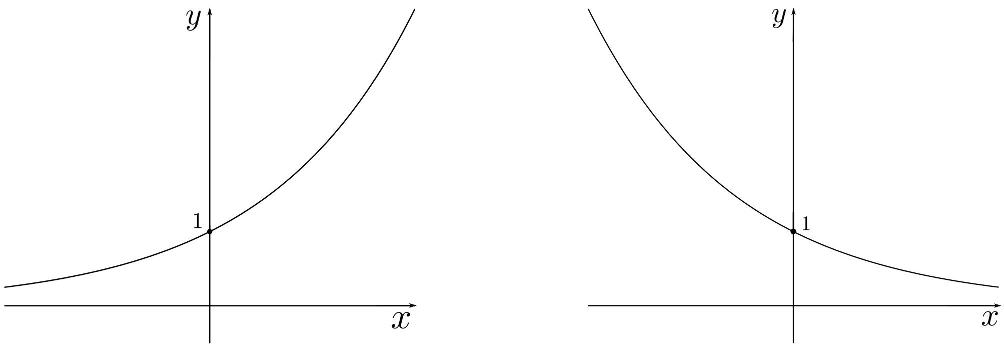
Finally, an exponential function can be defined by

$$f(x) = a^x, \quad x \in \mathbb{R},$$

where a is a positive constant. The domain of f is $\mathbb{R}$ and the range is $\mathbb{R}^+$.

If $a < 1$, it is common to define $b = \frac{1}{a}$. f can then be written as $f(x) = b^{-x}$ with $b > 1$.

The graph of f is as follows:



(a) $y = a^x, \quad a > 1.$ (b) $y = a^{-x}, \quad a > 1.$

Figure 1.5: Comparison of exponent graphs for different values of a.

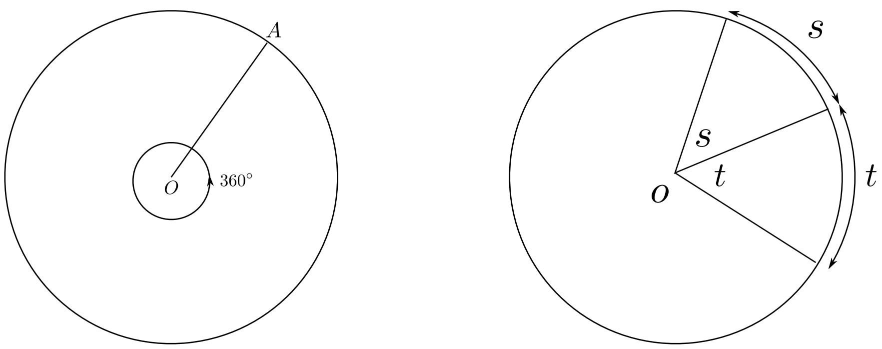
There is also a special exponential function, $f = e^x$, we will investigate this further later in the course.

1.6 Trigonometric functions

1.6.1 Measuring angles

Definition: Degrees
Degrees are defined so that one full turn is $360°$.

Definition: Radians
Radians (usually abbreviated as rad or ^{c}) are defined using a circle of radius 1. The angle between two radii in the unit circle in radians is equal to the arc length between the two radii.



(a) 360° is a full turn.

(b) A unit circle with angles of s^c and t^c marked.

We can see immediately that a full turn is 2π rad because a circle of radius 1 has a circumference 2π. Therefore we have

$$\begin{aligned} 1 \text{ turn} &= 360° = 2\pi \,\text{rad} \\ \tfrac{1}{2} \text{ turn} &= 180° = \pi \,\text{rad} \end{aligned}$$

So,

$$\begin{aligned} 1\,\text{rad} &= \frac{180°}{\pi} \\ 1° &= \frac{\pi}{180}\,\text{rad} \end{aligned}$$

If the radius is not 1, then you need to take the ratio

$$\frac{\text{arc length}}{\text{radius}} = \text{angle (in radians)}. \tag{1.11}$$

1.6.2 Trignometric functions: cosine, sine & tangent

Definition: sin, cos and tan
The values $\cos(\theta)$ and $\sin(\theta)$ (often written $\cos\theta$, $\sin\theta$) are the horizontal and vertical coordinates of the point C. $\tan(\theta)$ (often $\tan\theta$) is defined to be $\frac{\sin\theta}{\cos\theta}$.

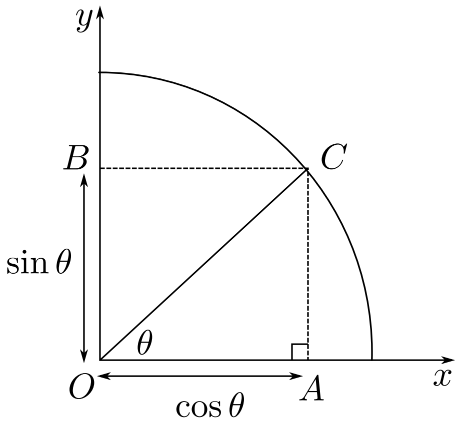


Figure 1.6: Geometric definition of cos and sin, circle radius $r = OC = 1$.

It can easily be seen that this definition is equivalent to the "SOH CAH TOA" definition you are familiar with:

$$\sin\theta = \frac{AC}{OC} = AC$$
$$\cos\theta = \frac{OA}{OC} = OA$$
$$\tan\theta = \frac{AC}{OA} = \frac{\sin\theta}{\cos\theta}$$

Although this definition allows for sin, cos and tan to easily be extended to angles outside the range $\left[0, \frac{\pi}{2}\right]$

1.6.3 Properties of sin, cos and tan

Property
$\cos^2\theta + \sin^2\theta = 1$

Proof: Use Pythagoras' Theorem in triangle OAC. □

Property
cos and sin are periodic functions with period 2π (i.e. for any x, $\cos(x+2\pi) = \cos x$, $\sin(x + 2\pi) = \sin x$).

Property
$\cos : \mathbb{R} \to [-1, 1]$ and $\sin : \mathbb{R} \to [-1, 1]$.

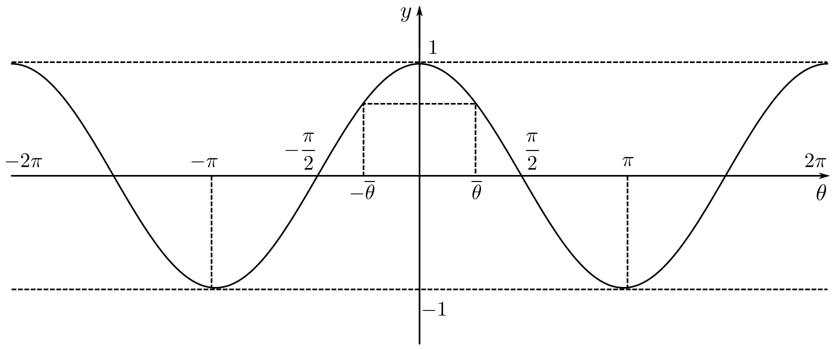


Figure 1.7: Graph of $\cos\theta$.

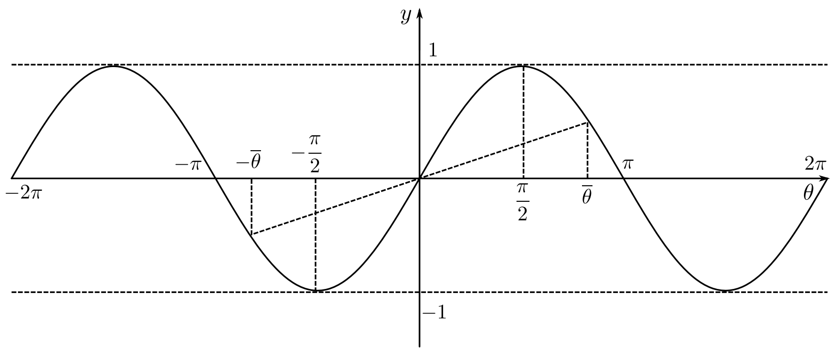


Figure 1.8: Graph of $\sin\theta$.

Property
cos is an even function. sin is an odd function.

Property
cos and sin are the same shape but shifted by $\pi/2$, which means

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$$

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

Property: Addition formulae

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

Property: Double angle formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$

Proof:

$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta) \\ &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta) \\ &= 2\sin\theta\cos\theta\end{aligned}$$

□

Property: Half angle formulae

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$
$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$$

Proof: Let $2\theta = \alpha$, then

$$\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 \implies \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$
$$\cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \implies \sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$$

□

Property
tan has vertical asymptotes at $\theta = \frac{\pi}{2}(2N-1)$ for $N \in \mathbb{Z}$.

Proof: At $\theta = \frac{\pi}{2}(2N-1)$, $\cos\theta = 0$.

□

Property
$\tan : \mathbb{R} \setminus \{\frac{\pi}{2}(2N-1) : N \in \mathbb{Z}\} \to \mathbb{R}$

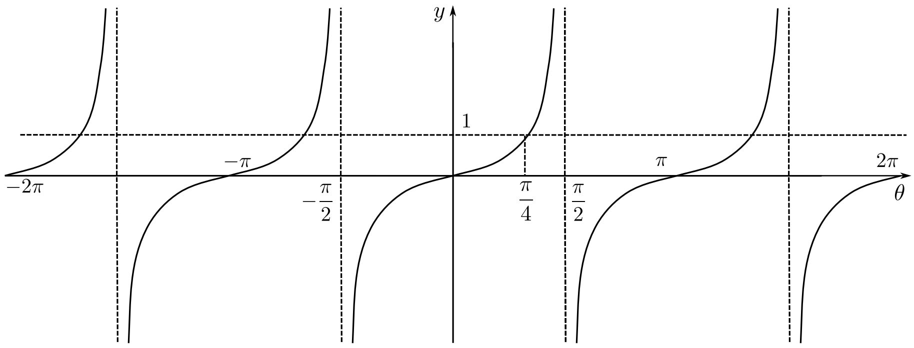


Figure 1.9: Graph of $\tan\theta$.

Property
tan is periodic with period π.

Property: Double angle formula for tan

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

Proof:

$$\begin{aligned}\tan(\theta+\phi) &= \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} \\ &= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}.\end{aligned}$$

Now divide by $\cos\theta\cos\phi$.

□

Definition
Secant, cosecant and cotangent The secant, cosecant and cotangent functions are defined as

$$\sec x = \frac{1}{\cos x}, \quad \text{cosec}x = \frac{1}{\sin x}, \quad \text{cot}x = \frac{1}{\tan x}.$$

Property

$$1 + \tan^2 x = \sec^2 x.$$

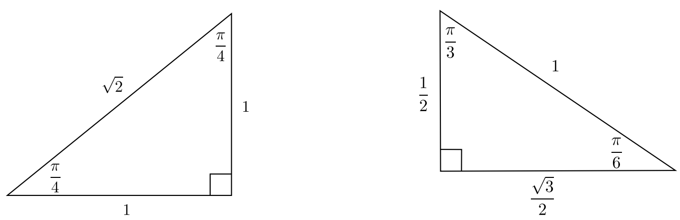
Proof: Divide $\cos^2\theta + \sin^2\theta = 1$ through by $\cos^2 x$. □

Property

There are a number of "special angles" for which you should remember the values of sin, cos and tan:

Angle (°)	**Angle (^{c})**	**sin**	**cos**	**tan**
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{3}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	∞

These can be easily remembered via the following triangles:



(a) $\pi/4$ triangle. (b) $\pi/3$, $\pi/6$ triangle.

Figure 1.10: Some well known results for particular angles can be derived by the above triangles for sin, cos and tan.

Chapter 2

Differentiation

2.1 Rates of change

Suppose we drive from UCL to Stratford-upon-Avon (100 miles). We plot a graph of the distance travelled against time. We want to measure how fast we traveled. The average speed of the trip is calculated as follows:

$$\frac{100 \text{ miles}}{2 \text{ hrs}} = 50 \text{ mph}.$$

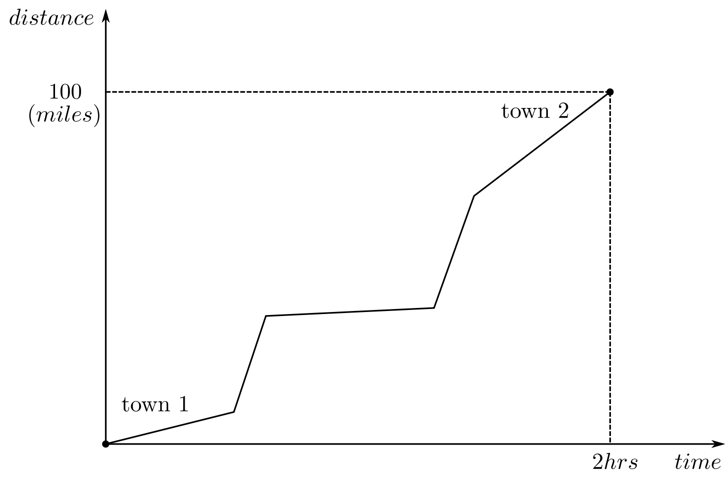


Figure 2.1: Graph showing distance travelled against time, from town 1 (UCL) to town 2 (Stratford-upon-Avon).

However, when travelling you do not stick to one speed, sometimes you do more than 50 mph, sometimes much less. The reading on your speedometer is your *instantaneous* speed. This corresponds to the *gradient* of the graph at the given point in your journey.

Definition
The **gradient** of a line is a measure of the steepness or slope of the line. It can be

found using:

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x}$$

The gradient of a curve is the gradient of the tangent at a given point.

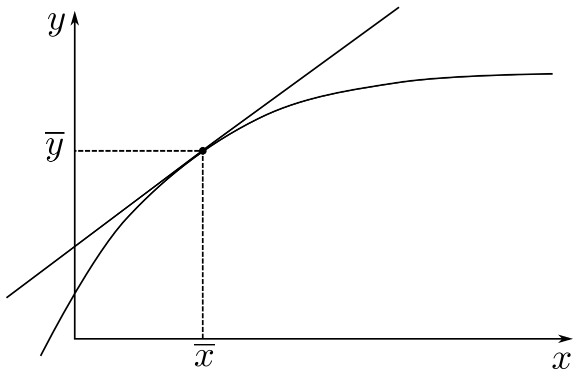


Figure 2.2: Curve $y = f(x)$ with tangent line at $(\overline{x}, \overline{y})$.

In the following section, we will be looking at methods for finding the gradients of graphs.

2.2 Finding the gradient

For mathematical curves, we will learn to find gradients algebraically.

To find the gradient of a curve $y = q(x)$ at $x = c$, we first consider the line joining the points $(c, q(c))$ $(c + h, q(c + h))$, where h is small.

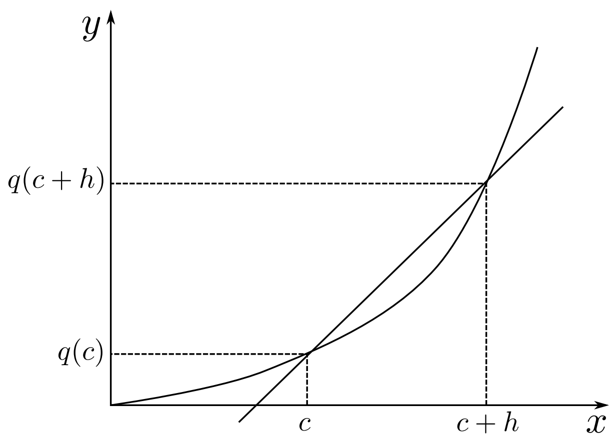


Figure 2.3: Graph showing line joining the points $(c, q(c))$ and $(c + h, q(c + h))$ on the curve $y = q(x)$.

We will look at the gradient of this line as we make h smaller and smaller, as this will get closer and closer to the gradient of the tangent.

Example

Let us start with the example of the curve $y = S(x) = x^2$.

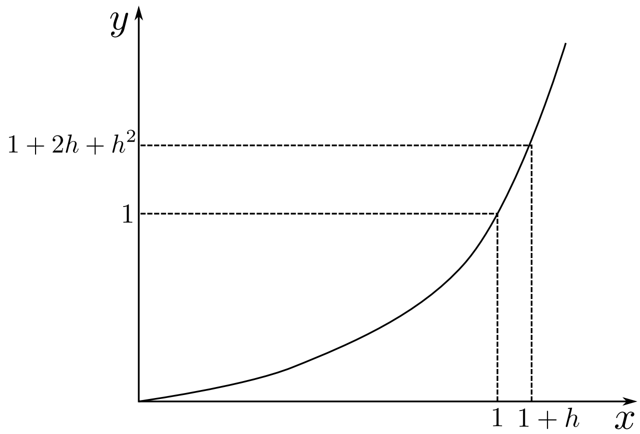


Figure 2.4: Curve $y = S(x) = x^2$ displaying small increment at $x = 1$.

Look at the point $(1, 1)$ on the curve. We want find the gradient at this point. Lets consider a line connecting $(1, 1)$ and $(1 + h, (1 + h)^2)$.
The gradient of this line is:

$$\frac{\text{change in } y}{\text{change in } x} = \frac{(1+h)^2 - 1}{1 + h - 1} \tag{2.1}$$

$$= \frac{h^2 + 2h}{h} \tag{2.2}$$

$$= h + 2 \tag{2.3}$$

To find the gradient at the point, we look at what will happen as $h \to 0$ (h tends to 0).

$$\text{As } h \to 0 \quad h + 2 \to 2$$

Therefore the gradient of the curve $y = x^2$ at the point $x = 1$ is 1.

We define the derivative as follows:

Definition

The **gradient** of $y = f(x)$ at x=c, written $\frac{dy}{dx}$ at c or $f'(c)$, is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

$\lim_{h\to 0}$ is the limit as h gets closer and closer to 0. This definition is exactly what we used in the example.

If we leave c as a variable instead of subsituting in a value, we can find the gradient of the whole curve.

Example
Let us consider the function $q(x) = x^3$. At $x = c + h$ we have

$$q(c+h) = (c+h)^3 = c^3 + 3c^2h + 3ch^2 + h^3.$$

Therefore,

$$q'(c) = \lim_{h \to 0} \frac{c^3 + 3c^2h + 3ch^2 + h^3 - c^3}{h} \tag{2.4}$$

$$= \lim_{h \to 0} \frac{3c^2h + 3ch^2 + h^3}{h} \tag{2.5}$$

$$= \lim_{h \to 0} 3c^2 + 3ch + h^2 \tag{2.6}$$

$$= 3c^2 \tag{2.7}$$

or in other words,

$$q'(x) = 3x^2.$$

Example
Now let us consider the function $r(x) = 1/x$. In this case we have

$$r(c+h) - r(c) = \frac{1}{c+h} - \frac{1}{c}.$$

Now, let us consider the ratio

$$\frac{r(c+h) - r(c)}{h} = \frac{1}{h}\left(\frac{1}{c+h} - \frac{1}{c}\right) = \frac{1}{h}\left(\frac{-h}{c(c+h)}\right) = -\frac{1}{c(c+h)},$$

and as $h \to 0$, we have

$$r'(c) = -\frac{1}{c^2}, \quad \text{i.e.} \quad r'(x) = -\frac{1}{x^2}, \quad x \neq 0.$$

note: $r(x) = 1/x$ is not well defined at $x = 0$ and in this case, nor is its derivative.

Definition
If the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, the function f is **differentiable**.

Example
An example of a function which is not differentiable at a certain point:

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

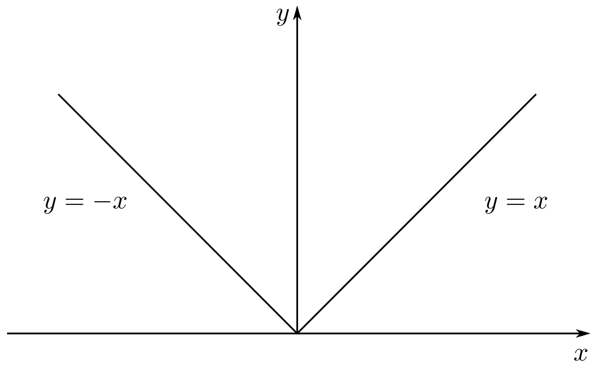


Figure 2.5: *Graph of* $y = |x|$.

At $x = 0$, $f(x)$ is continuous but not differentiable, since through the point $(0, 0)$, you can draw many, many tangent lines. We can also show

$$\text{for} \quad h > 0, \quad \frac{f(0+h) - f(0)}{h} = \frac{h - 0}{h} = 1,$$

$$\text{for} \quad h < 0, \quad \frac{f(0+h) - f(0)}{h} = \frac{-h - 0}{h} = -1,$$

i.e.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

doesn't exist! Taking the limit from both sides must give the same answer.

2.3 Some common derivatives

The derivative of x^n is nx^{n-1}.

Proof:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n\} - x^n}{h} \\
&= \lim_{h \to 0} \left\{ nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} \right\} \\
&= nx^{n-1}.
\end{aligned}$$

□

The derivative of $\sin x$ is $\cos x$.

note: For this to be true, x must be measured in radians.

Proof:

$$\begin{aligned} \frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\ & \quad \text{When } h \text{ is small, } \sin h \approx h \text{ and } \cos h \approx 1\text{, so:} \\ &= \lim_{h\to 0} \frac{\sin x+h\cos x-\sin x}{h} \\ &= \lim_{h\to 0} \cos x \\ &= \cos x \end{aligned}$$

□

The derivative of $\cos x$ is $-\sin x$.

note: For this to be true, x must be measured in radians.

Proof:

$$\begin{aligned} \frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x-h\sin x-\cos x}{h} \\ &= \lim_{h\to 0} -\sin x \\ &= -\sin x \end{aligned}$$

□

The derivative of $\tan x$ is $\sec^2 x$.
We will see why this is true later.

We could continue working through all the functions we would like to differentiate and working out their derivatives, but this take a long time. Instead, there are some rules which we can use to save time.

2.4 Rules for differentiation

There are three key rules we can use to differentiate more complicated functions.

2.4.1 The sum rule

The sum rule
If f and g are differentiable, then

$$\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{d}{dx}\left(f(x)\right)+\frac{d}{dx}\left(g(x)\right)$$

or

$$\frac{d}{dx}\left(f(x)+g(x)\right)=f'(x)+g'(x)$$

Example
Consider the function $f(x)=\left(x^3+x^4\right)$, then using the above we have

$$\begin{aligned}\frac{d}{dx}\left(x^3+x^4\right)&=\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^4\right)\\&=3x^2+4x^3.\end{aligned}$$

If you repeatedly apply the sum rule, you have

$$\frac{d}{dx}\left(f_1(x)+f_2(x)+\cdots+f_n(x)\right)=\frac{d}{dx}\left(f_1(x)\right)+\frac{d}{dx}\left(f_2(x)\right)+\cdots+\frac{d}{dx}\left(f_n(x)\right).$$

2.4.2 The product rule

The product rule
If f and g are differentiable, then

$$\frac{d}{dx}\left(f(x)g(x)\right)=\frac{d}{dx}\left(f(x)\right)g(x)+f(x)\frac{d}{dx}\left(g(x)\right)$$

or

$$\frac{d}{dx}\left(f(x)g(x)\right)=f'(x)g(x)+f(x)g'(x)$$

Example

$$\begin{aligned}\frac{d}{dx}\left[(x^2+1)(x^3-1)\right]&=2x(x^3-1)+(x^2+1)3x^2\\&=2x^4-2x+3x^4+3x^2\\&=5x^4+3x^2-2x.\end{aligned}$$

Here we have put

$$f(x) = x^2 + 1 \quad \Longrightarrow \quad f'(x) = 2x,$$

and

$$g(x) = x^3 - 1 \quad \Longrightarrow \quad g'(x) = 3x^2.$$

Example
Consider the derivative of x^5, so

$$\begin{aligned} \frac{d}{dx}\left(x^5\right) &= \frac{d}{dx}\left(x^4 \cdot x\right) \\ &= 4x^3 \cdot x + x^4 \cdot 1 \\ &= 5x^4, \end{aligned}$$

as expected, since

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

Here we have put

$$f(x) = x^4 \quad \Longrightarrow \quad f'(x) = 4x^3,$$

and

$$g(x) = x \quad \Longrightarrow \quad g'(x) = 1.$$

2.4.3 The chain rule

The chain rule
If f and g are differentiable, then

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$$

Example
Consider the function $y(x) = (x^3 + 2x)^{10}$. Here we will choose $f(w) = w^{10}$ and $g(x) = x^3 + 2x$, (so $f'(w) = 10w^9$ and $g'(x) = 3x^2 + 2$). Then

$$\begin{aligned} \frac{d}{dx}\left(y(x)\right) &= \frac{d}{dx}\left((x^3 + 2x)^{10}\right) \\ &= \frac{d}{dx}\left(f(g(x))\right) \\ &= f'(g(x))g'(x) \\ &= 10(x^3 + 2x)^9 \cdot (3x^2 + 2). \end{aligned}$$

Essentially, what we have done is to substitute $g(x) = x^3 + 2x$ in our function for $y(x)$, to make the differentiation easier.

Example
To find

$$\frac{d}{dx}\left(\sin(1+x^2)\right)$$

take $f(w) = \sin w$ and $g(x) = 1 + x^2$ ($f'(w) = \cos w$ and $g'(x) = 2x$).
Then

$$\begin{aligned}\frac{d}{dx}\left(\sin(1+x^2)\right) &= \frac{d}{dx}\left(f(g(x))\right)\\ &= f'(g(x))g'(x)\\ &= \cos(1+x^2)\cdot 2x\\ &= 2x\cos(1+x^2)\end{aligned}$$

2.4.4 Some more difficult examples

Example
To differentiate

$$\sin\left(\sqrt{1+x^2}\right)$$

we must use the chain rule twice.

$$\begin{aligned}\frac{d}{dx}\left(\sin\left(\sqrt{1+x^2}\right)\right) &= \frac{d}{dx}\left(\sin\left((1+x^2)^{\frac{1}{2}}\right)\right)\\ &= \cos\left((1+x^2)^{\frac{1}{2}}\right)\cdot\frac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right)\\ &= \cos\left((1+x^2)^{\frac{1}{2}}\right)\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\frac{d}{dx}\left(1+x^2\right)\\ &= \cos\left((1+x^2)^{\frac{1}{2}}\right)\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x\\ &= \frac{x\cos\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\end{aligned}$$

Example
To differentiate

$$(x^2+3)\sin x\cos x$$

we must use the product rule twice.

$$\begin{aligned}\frac{d}{dx}\left((x^2+3)\sin x\cos x\right) &= (x^2+3)\frac{d}{dx}\left(\sin x\cos x\right) + \sin x\cos x\frac{d}{dx}\left(x^2+3\right)\\ &= (x^2+3)\left(\sin x\frac{d}{dx}\left(\cos x\right) + \cos x\frac{d}{dx}\left(\sin x\right)\right) + 2x\sin x\cos x\\ &= (x^2+3)\left(-\sin^2 x + \cos^2 x\right) + 2x\sin x\cos x\end{aligned}$$

Example
To differentiate

$$\sqrt{x^2-3}\sin x$$

we must use the product rule and the chain rule.

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{x^2-3}\sin x\right) &= \frac{d}{dx}\left((x^2-3)^{\frac{1}{2}}\sin x\right) \\
&= \sin x\frac{d}{dx}\left((x^2-3)^{\frac{1}{2}}\right) + (x^2-3)^{\frac{1}{2}}\frac{d}{dx}(\sin x) \\
&= \sin x\cdot\frac{1}{2}(x^2-3)^{-\frac{1}{2}}\cdot 2x + (x^2-3)^{\frac{1}{2}}\cos x \\
&= \frac{\sin x}{\sqrt{x^2-3}} + \sqrt{x^2-3}\cos x
\end{aligned}$$

Example
To differentiate

$$\tan x$$

we can use the product rule and the chain rule.

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{d}{dx}\left(\sin x(\cos x)^{-1}\right) \\
&= \sin x\frac{d}{dx}\left((\cos x)^{-1}\right) + (\cos x)^{-1}\frac{d}{dx}(\sin x) \\
&= \sin x\cdot -(\cos x)^{-2}\cdot -\sin x + (\cos x)^{-1}\cos x \\
&= \frac{\sin^2 x}{\cos^2 x} + \frac{\cos x}{\cos x} \\
&= \tan^2 x + 1 \qquad\qquad = \sec^2 x
\end{aligned}$$

This is one way of showing that the derivative of $\tan x$ is $\sec^2 x$.

2.4.5 The quotient rule

The quotient rule is a special case of the product rule. It is up to you whether you learn and use the quotient rule or whether you use the product rule instead.

The quotient rule
If f and g are differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

Proof: Apply the product and chain rules to $f(x)(g(x))^{-1}$

□

Example
To differentiate

$$\tan x$$

we can use the quotient rule.

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x \cos x - \sin x \cdot - \sin x}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

This is another way of showing that the derivative of $\tan x$ is $\sec^2 x$.

2.5 Uses of differentiation

2.5.1 Finding the gradient at a point

To find the gradient of a curve at a given x co-ordinate, simply substitute the value of x into the derivative.

Example
To find the gradient of $y(x) = x^3 - x^2$ at $x = 3$, first find $y'(x)$:

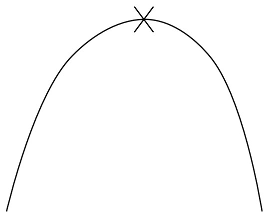
$$y'(x) = 3x^2 - 2x$$

Next subsitute $x = 3$:

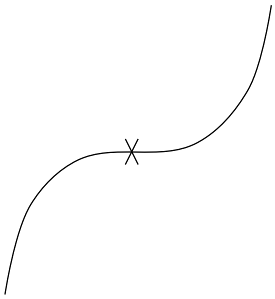
$$\begin{aligned}y'(3) &= 3 \cdot 3^2 - 2 \cdot 3\\ &= 21\end{aligned}$$

2.5.2 Finding the maximum and minimum points

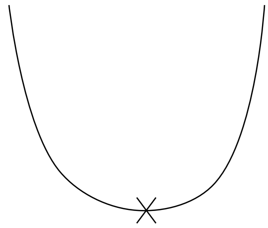
At a point where $\frac{dy}{dx} = 0$, there are three possibilites:



(a) *Local maximum.*



(b) *Point of inflection.*



(c) *Local minimum.*

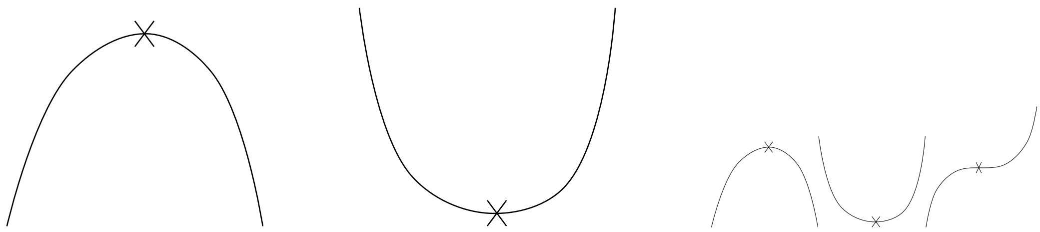
Figure 2.6: Different options for when $\frac{dy}{dx} = 0$.

In order to tell which of these occurs at a given point, we must look at the second derivative.

Definition
The **second derivative** of a function, written $f''(x)$ or $\frac{d^2f}{dx^2}$ is obtained by differentiating $\frac{dy}{dx}$.

Example
If
$$f(x) = x^3$$
then
$$f'(x) = 3x^2$$
and
$$f''(x) = 6x.$$

The second derivative gives that rate at which the gradient is changing. If the gradient is increasing at a turning point, then the point is a minimum. Similarly, if the gradient is decreasing at a turning point, then the point is a maximum. If the second derivative is 0 at the turning point, we need more information.



(a) Local maximum

$f'(x) = 0$

$f''(x) < 0$

(b) Local minimum

$f'(x) = 0$

$f''(x) > 0$

(c) Need more information

$f'(x) = 0$

$f''(x) = 0$

Figure 2.7: Different options for when $\frac{dy}{dx} = 0$.

Example
Let $f(x) = x^2$.
$f'(x) = 2x$, so f has a turning point at $x = 0$.
$f''(x) = 2$, so $f''(0) > 0$. Therefore the turning point is a minimum

Example
Let $f(x) = x^3$.
$f'(x) = 3x^2$, so f has a turning point at $x = 0$.
$f''(x) = 6x$, so $f''(0) = 0$. We need more information to decide what happens at this point.
In this case, at $x = 0$ there is a point of inflection.

Example
Let $f(x) = x^4$.
$f'(x) = 4x^3$, so f has a turning point at $x = 0$.
$f''(x) = 12x^2$, so $f''(0) = 0$. We need more information to decide what happens at this point.
In this case, at $x = 0$ there is a minimum.

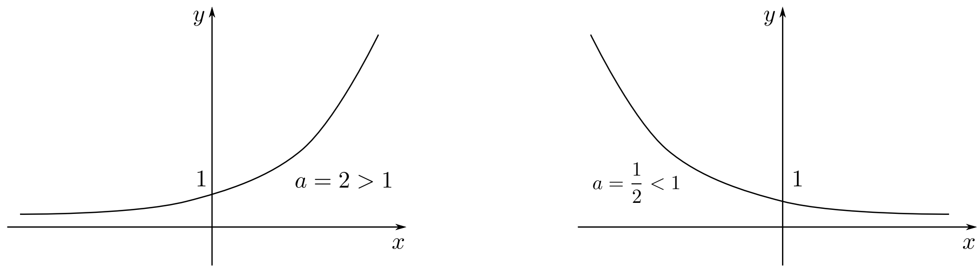
2.6 Exponentials & Logarithms

An exponential function is a function of the form

$$f(x) = a^x,$$

where a is a positive constant.

Example



(a) $y = 2^x$ (b) $y = \left(\frac{1}{2}\right)^x$

Figure 2.8: Two exponential functions

Properties

If $a > 1$, $f(x)$ increases as x increases.

If $a < 1$, $f(x)$ decreases as x increases.

If $a = 1$, $f(x) = 1$.

$a^0 = 1$ for each a, so the graph always passes through the point $(0, 1)$.

2.6.1 The gradient of e^x

First let us consider the gradient of a^x at $x = 0$.

Example
Suppose we have $f(x) = 2^x$, then applying the definition of the derivative we have

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{2^h - 1}{h}.$$

h	$f'(0) \approx$
0.1	0.7177
0.01	0.6955
0.001	0.6933
0.0001	0.6932

So for $f(x) = 2^x$, $(a = 2)$, we have slope ≈ 0.693 at $x = 0$.
Similarly, for $f(x) = 3^x$, $(a = 3)$, we have slope ≈ 1.698 at $x = 0$.

Therefore, we expect that there is a number between 2 and 3 such that the slope at $x = 0$ is 1. This number is called e, where $e \approx 2.718281828459\ldots$. The number e is irrational.

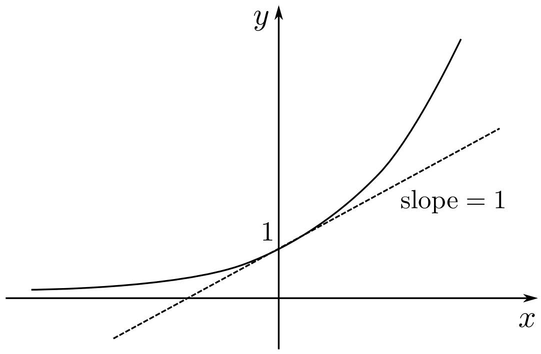


Figure 2.9: *Graph of $y = e^x$, which has a gradient of 1 at $x = 0$.*

The fact that the slope is 1 at $x = 0$ tells us that

$$\frac{e^h - e^0}{h} = \frac{e^h - 1}{h} \to 1, \quad \text{as } h \to 0.$$

Definition
We call $f(x) = e^x = \exp(x)$ the **exponential function**.

Property

$$\frac{d}{dx}(e^x) = e^x.$$

Proof: To find the gradient at $x = c$, we need to look at

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{e^{c+h} - e^c}{h} \\ &= \lim_{h\to 0} \frac{e^c(e^h - 1)}{h} \\ &= e^c \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^c, \end{aligned}$$

□

Example
To find

$$\frac{d}{dx}\left(e^{\sqrt{1+x}}\right)$$

we must use the chain rule. Choose $g(x) = \sqrt{1+x}$ and $f(u) = e^u$, so we have

$g'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ and $f'(u) = e^u$.

$$\begin{aligned} \frac{d}{dx}\left(e^{\sqrt{1+x}}\right) &= f'(g(x))g'(x) \\ &= e^{\sqrt{1+x}} \cdot \frac{1}{2}(1+x)^{-\frac{1}{2}} \\ &= \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}}. \end{aligned}$$

2.6.2 Logarithms

Definition
The inverse of an exponenetial function is called a **logarithm** or **log**.
The inverse of a^x is written as $\log_a x$ and is defined such that

$$a^{\log_a x} = x.$$

Example
$\log_{10}(1000) = 3$ because $10^3 = 1000$.
$\log_2(16) = 4$ because $2^4 = 16$.
$\log_{10}(2) = 0.301...$ because $10^{0.301...} = 2$.

Laws of Logs
The following properties hold for logarithms:

1. $\log_a(MN) = \log_a M + \log_a N$.
2. $\log_a(M^p) = p \log_a M$.

Logarithms are used among other things to solve exponential equations.

Example
Find x, given $3^x = 7$. Taking the logarithm of both sides we have

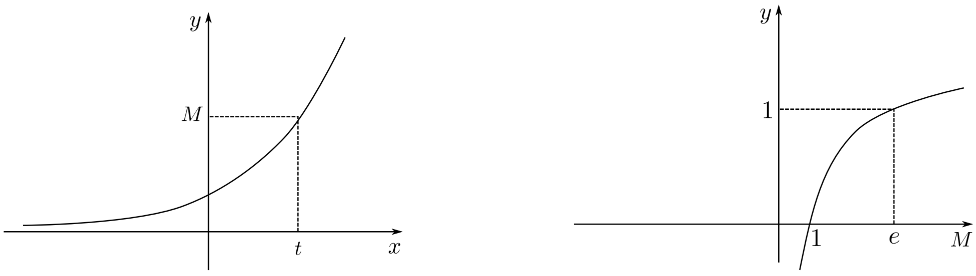
$$\ln(3^x) = \ln 7 \quad \Longrightarrow \quad x \ln 3 = \ln 7.$$

Rearranging we have

$$x = \frac{\ln 3}{\ln 7} \approx \frac{1.95}{1.10} \approx 1.77.$$

The natural logarithm

Definition
The inverse of $f(x) = e^x$ is called the **natural logarithm** and is written $\ln x$.



(a) $y = e^x$.

(b) $y = \ln x$.

Figure 2.10: Graphs of the exponential functions and the natural logarithm.

2.6.3 Differentiation of other exponentials

Example
In order to differentiate 3^x, we must express it in terms of e^x:

$$3 = e^{\ln 3} \quad \Longrightarrow \quad 3^x = (e^{\ln 3})^x = e^{x \ln 3}$$

Therefore we calculate the derivative of 3^x as follows:

$$\begin{aligned} \frac{d}{dx}(3^x) &= \frac{d}{dx}(e^{x \ln 3}) \\ &= e^{x \ln 3} \cdot \ln 3 \\ &= 3^x \cdot \ln 3. \end{aligned}$$

In general, for any positive constant a

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

2.7 Differentiating inverse functions

Definition
If a function f is one-to-one, we can find its **inverse**, f^{-1}. The inverse satisfies

$$f^{-1}(f(x)) = x$$

for all values of x in the domain of f.

This notation is most commonly used for trigonometric functions ($\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.).

note: $\tan^{-1} x$ is used for the inverse of tan and NOT $\frac{1}{\tan x}$.

note: Sometimes, arcsin, arccos and arctan are used to represent $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Finding the derivative of an inverse
Let f be a function. The derivative of f^{-1} is:

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

Proof: Let f be a function and let $g = f^{-1}$. By the chain rule,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

g is f^{-1}, so $f(g(x)) = x$ and

$$\frac{d}{dx} f(g(x)) = 1.$$

Therefore,

$$1 = f'(g(x))g'(x).$$

Rearranging gives

$$g'(x) = \frac{1}{f'(g(x))}.$$

□

The method in the proof can be used to differentiate inverse functions:

Example
To find

$$\frac{d}{dx}\left(\sin^{-1} x\right),$$

we first look at

$$\begin{aligned}\frac{d}{dx}\left(\sin\left(\sin^{-1} x\right)\right) &= \frac{d}{dx}(x)\\ &= 1.\end{aligned}$$

Using the chain rule,

$$\frac{d}{dx}\left(\sin\left(\sin^{-1}x\right)\right) = \cos\left(\sin^{-1}x\right)\cdot\frac{d}{dx}\left(\sin^{-1}x\right).$$

Therefore:

$$\cos\left(\sin^{-1}x\right)\cdot\frac{d}{dx}\left(\sin^{-1}x\right) = 1$$
$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\cos\left(\sin^{-1}x\right)}$$

We can simplify this, by letting $\theta = \sin^{-1}x$, then looking at the following triangle:

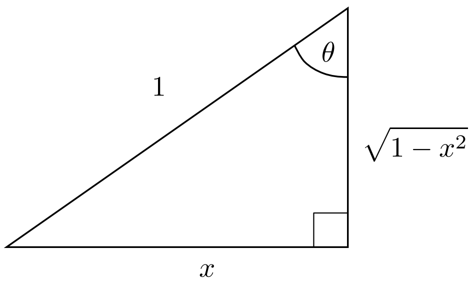


Figure 2.11: $\sin(\theta) = x$

This tells us that $\cos\theta = \sqrt{1-x^2}$. Therefore

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}.$$

We can also find the derivatives of inverses by using:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Example
To find

$$\frac{d}{dx}\left(\sin^{-1}x\right),$$

let $y = \sin^{-1}x$. This means that $x = \sin y$ and so:

$$\frac{dx}{dy} = \cos y$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}}\\ &= \frac{1}{\cos y}\\ &= \frac{1}{\cos\left(\sin^{-1}x\right)}\end{aligned}$$

Simplifying as before gives

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}.$$

Example

$$\begin{aligned}\frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sin(\cos^{-1} x)}\\ &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Example

$$\begin{aligned}\frac{d}{dx}\left(\tan^{-1} x\right) &= \frac{1}{1+\tan^2(\tan^{-1} x)}\\ &= \frac{1}{1+x^2}\end{aligned}$$

2.7.1 Differentiating Logarithms

Property

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example
To find

$$\frac{d}{dx}\left(\ln(\cos x)\right)$$

we must use the chain rule. Choose $g(x) = \cos x$ and $f(u) = \ln u$, so we have $g'(x) = -\sin x$ and $f'(u) = 1/u$. Thus

$$\begin{aligned}\frac{d}{dx}(\ln(\cos x)) &= f'(g(x))g'(x)\\ &= \frac{1}{\cos x}\cdot(-\sin x)\\ &= -\tan x.\end{aligned}$$

Example

$$\begin{aligned}\frac{d}{dx}(\sin(\ln x)) &= \cos(\ln x)\cdot\frac{1}{x}\\ &= \frac{\cos(\ln x)}{x}.\end{aligned}$$

Property: Change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular,

$$\log_a x = \frac{\ln x}{\ln a}.$$

Proof: Let

$$m = \log_a x.$$

This means that

$$a^m = x.$$

Applying $\log_b$ to both sides gives:

$$\log_b a^m = \log_b x$$

$$m \log_b a = \log_b x$$

$$m = \frac{\log_b x}{\log_b a}$$

□

Property

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}.$$

Proof:

$$\begin{aligned}\frac{d}{dx}\left(\log_a x\right) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right)\\ &= \frac{1}{x\ln a}\end{aligned}$$

□

2.8 Polar co-ordinates

A circle with radius r can be written as $x^2+y^2=r^2$. However, it is more natural to think of a circle as $r=$ constant.

We can write a circle in this more natural way by using polar co-ordinates.

Definition
In **polar co-ordinates**, we give the distance from the origin, r, and the angle made with the x-axis, *theta*, as shown on the diagram below.

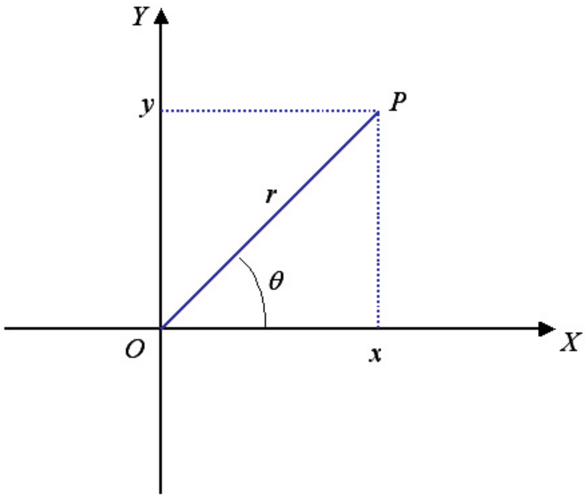


Figure 2.12: Polar co-ordinates are given by r and θ.

Shapes can be defined by writing r in terms of θ.

The Cartesian (normal) co-ordinates can be written in terms of the polar co-ordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Example
The following cardioid can be most easily written in polar co-ordinates as $r = 1-\cos\theta$.

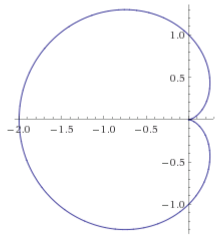


Figure 2.13: The cardioid $r = 1 - \cos\theta$.

If written in Cartesian co-ordinates, it would be

$$x^2 + y^2 + x = \sqrt{x^2 + y^2}.$$

2.8.1 Implicit Differentiation

The chain rule can be rearranged to give:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

This can be used whenever y and x are known in terms of a parameter t.

Example
If $x = t^2 + 4$ and $y = e^t$ then:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{e^t}{2t} \end{aligned}$$

In most cases, the answer can be left in terms of t.

Example
To find the gradient of the cardioid given by the polar equation $r = 1 - \cos\theta$, we must first write x and y in terms of θ.

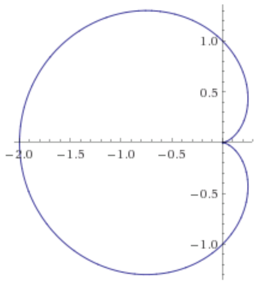


Figure 2.14: The cardioid with polar equation $r = 1 - \cos\theta$.

$$\begin{aligned} x &= r\cos\theta \\ &= (1-\cos\theta)\cos\theta \\ &= \cos\theta - \cos^2\theta \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= (1-\cos\theta)\sin\theta \\ &= \sin\theta - \sin\theta\cos\theta \end{aligned}$$

Now we can differentiate:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\cos\theta + \sin^2\theta - \cos^2\theta}{-\sin\theta + 2\cos\theta\sin\theta} \end{aligned}$$

2.9 Real life examples

Example
You are required to design an open box with a square base and a total volume of 4m^3 using the least amount of materials.
We are hunting for particular dimensions of the box. Lets give them names, so we will call the length of the base edges a and let the height of the box be called b.

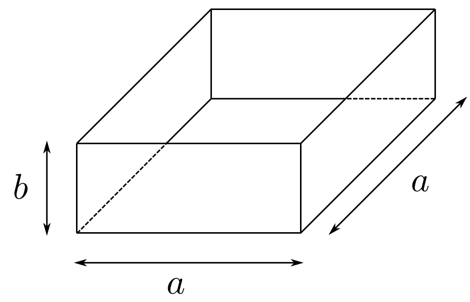


Figure 2.15: Square based open top box.

In terms of a and b, the volume of the box is

$$a^2b = 4 \ (\text{m}^3).$$

We want to minimise the amount of material, so we must look at how much material is used. We have a base of area a^2 and 4 sides, each with area ab. So the total area is

$$a^2 + 4ab.$$

We want to find the minimum value of $a^2 + 4ab$, subject to the condition $a^2b = 4$ and more obviously $a > 0$, $b > 0$. From the condition, we see that $b = 4/a^2$, so we can rewrite the amount of material in terms of a:

$$a^2 + 4ab = a^2 + 4a\left(\frac{4}{a^2}\right) = a^2 + \frac{16}{a}.$$

So now we can write a function for the material area solely in terms of one of the lengths of the box (the other is now fixed by using the condition). That is

$$f(a) = a^2 + \frac{16}{a}.$$

Now written like a function, it is easy to see how we would minimise the material, that is by minimising the function $f(a)$. So we have to calculate $f'(a)$, which is

$$f'(a) = 2a - \frac{16}{a^2},$$

and now we simply need to see when $f'(a) = 0$:

$$2a - \frac{16}{a^2} = 0 \quad \Longrightarrow \quad 2a^3 = 16 \quad \Longrightarrow \quad a = 2.$$

The only real number at which $f'(a) = 0$ is $a = 2$. The function $f'(a)$ makes sense except at $a = 0$, which is outside the range (since $a > 0$). Differentiating $f'(a)$ gives

$$f''(a) = 2 + \frac{32}{a^3}.$$

This is positive for the whole domain (and at $a = 2$), and so this point is a minimum.

Example

We have a pair of islands, island 1 and island 2, 20km and 10km away from a straight shore, respectively. The perpendiculars from the islands to the shore are 30km apart (along the shore). What is the quickest way between the two islands that goes via the shore?

We are trying to find a point along the shore, which we want to visit when going from one island to the other. This point can specified by the distance from the perpendicular of island 1, call this distance x.

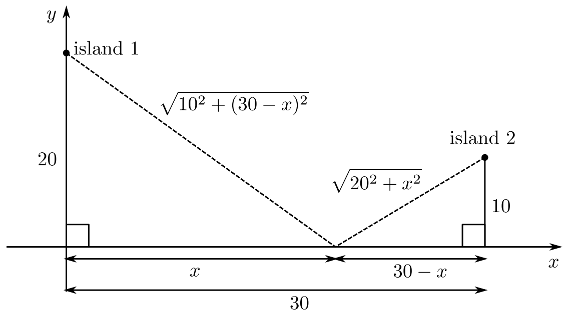


Figure 2.16: The set up of islands relative to the shore (x-axis).

The total distance $D(x)$ to be travelled is given by the formula

$$D(x) = \sqrt{20^2 + x^2} + \sqrt{10^2 + (30 - x)^2}.$$

On geometric grounds, we see that $0 \leq x \leq 30$, and $D(x)$ is differentiable on this domain.

$$\begin{aligned} D'(x) &= \frac{1}{2}(20^2 + x^2)^{-\frac{1}{2}} \cdot 2x + \frac{1}{2}\left(10^2 + (30 - x)^2\right)^{-\frac{1}{2}} 2(30 - x) \cdot -1 \\ &= \frac{x}{\sqrt{20^2 + x^2}} - \frac{30 - x}{\sqrt{(30 - x)^2 + 10^2}}. \end{aligned}$$

To find the minimum distance we set $D'(x) = 0$, so we have

$$\frac{x}{\sqrt{20^2 + x^2}} = \frac{30 - x}{\sqrt{(30 - x)^2 + 10^2}}.$$

Squaring both sides we get

$$\frac{x^2}{20^2 + x^2} = \frac{(30 - x)^2}{(30 - x)^2 + 10^2} \quad \Longrightarrow \quad (x^2)[(30 - x)^2 + 10^2] = (30 - x)^2(20^2 + x^2)$$

Expanding the brackets we see

$$10^2 x^2 = 20^2 (30 - x)^2.$$

Take the square root of both sides of the last equation gives

$$10x = \pm 20(30 - x).$$

Since $30 - x \geq 0$ and $x \geq 0$, the negative sign is not possible, so

$$10x = 20(30 - x) \quad \Longrightarrow \quad x = 20.$$

The only possible places where minimum can occur are

$$x = 0, \quad x = 20, \quad x = 30,$$

where

$$D(0) = 20 + \sqrt{10^2 + 30^2} \approx 51.6,$$
$$D(20) = 20\sqrt{2} + 10\sqrt{2} = 30\sqrt{2} \approx 42.4,$$
$$D(30) = \sqrt{20^2 + 30^2} + 10 \approx 46.06.$$

So the minimum value does occur at $x = 20$.

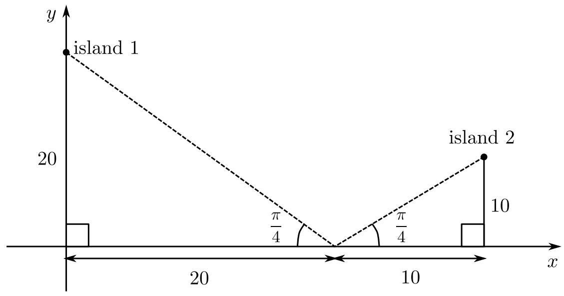


Figure 2.17: Optimum route from island 1 to island 2, arriving and leaving shore at an angle of $\pi/4$ radians.

The optimal route is to leave the shore at the same angle of arrival.
Could we have deduced that this route was the shortest without calculus? Yes! We could have reflected island 2 in the shore line to obtain an imaginary island. Then it is easy to see that the shortest route from island 1 to the imaginary island is a straight line.

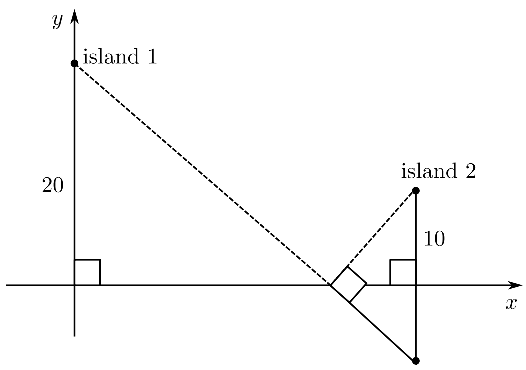


Figure 2.18: Sketch of alternative method of calculating shortest distance between the islands, using simple geometric properties, gaining same result.

2.9.1 Exponential growth and decay

Let $y = f(t)$ represent some physical quantity, such as the volume of a substance, the population of a certain species or the mass of a decaying radioactive substance. We want to measure the growth or decay of $f(t)$.

In many applications, the rate of growth (or decay) of a quantity is proportional to the quantity. In other "words":

$$\frac{dy}{dt} = \alpha y, \quad \alpha = \text{constant}.$$

This is a **differential equation** whose solution is

$$y(t) = ce^{\alpha t},$$

where constant c is determined by an **initial condition**, say, $y(0) = y_0$ (given). Therefore we have

$$y(t) = y_0 e^{\alpha t}.$$

This means that if you start with y_0, after time t you have $y(t)$.

If $\alpha > 0$, the quantity is increasing (growth). If $\alpha < 0$, the quantity is decreasing (decay).

We will study differential equations in more detail later in the course.

Radioactive decay

Atoms of elements which have the same number of protons but differing numbers of neutrons are referred to as isotopes of each other. Radioisotopes are isotopes that decompose and in doing so emit harmful particles and/or radiation.

It has been found experimentally that the atomic nuclei of so-called radioactive elements spontaneously decay. They do it at a characteristic rate.

If we start with an amount M_0 of an element with decay rate λ (where $\lambda > 0$), then after time t, the amount remaining is

$$M = M_0 e^{-\lambda t}.$$

This is the radioactive decay equation. The proportion left after time t is

$$\frac{M}{M_0} = e^{-\lambda t},$$

and the proportion decayed is

$$1 - \frac{M}{M_0} = 1 - e^{-\lambda t},$$

Carbon dating

Carbon dating is a technique used by archeologists and others who want to estimate the age of certain artefacts and fossils they uncover. The technique is based on certain properties of the carbon atom.

In its natural state, the nucleus of the carbon atom C^{12} has 6 protons and 6 neutrons. The isotope carbon-14, C^{14}, has 6 protons and 8 neutrons and is radioactive. It decays by beta emission.

Living plants and animals do not distinguish between C^{12} and C^{14}, so at the time of death, the ratio C^{12} to C^{14} in an organism is the same as the ratio in the atmosphere. However, this ratio changes after death, since C^{14} is converted into C^{12} but no further C^{14} is taken in.

Example
Half-lives: how long before half of what you start with has decayed? When do we get $M = \frac{1}{2}M_0$? We need to solve

$$\frac{M}{M_0} = \frac{1}{2} = e^{-\lambda t},$$

taking logarithms of both sides gives

$$\ln\left(\frac{1}{2}\right) = -\lambda t \qquad \Longrightarrow \qquad t = \frac{\ln(2)}{\lambda}.$$

So, the half-life, $T_{\frac{1}{2}}$ is given by

$$T_{\frac{1}{2}} = \frac{\ln(2)}{\lambda}.$$

If λ is in "per year", then $T_{\frac{1}{2}}$ is in years.

Example
Carbon-14 (C^{14}) exists in plants and animals, and is used to estimate the age of certain fossils uncovered. It is also used to trade metabolic pathways. C^{14} is radioactive and has a decay rate of $\lambda = 0.000125$ (per year). So we can calculate its half-life as

$$T_{\frac{1}{2}} = \frac{\ln 2}{0.000125} \approx 5545 \text{ years}.$$

Example
A certain element has $T_{\frac{1}{2}}$ of 10^6 years

1. What is the decay rate?

$$\lambda = \frac{\ln 2}{T_{\frac{1}{2}}} \approx \frac{0.693}{10^3} \approx 7 \times 10^{-7} \text{ (per year)}.$$

2. How much of this will have decayed after 1000 years? The proportion remaining is

$$\frac{M}{M_0} = e^{-\lambda t} = e^{-7\times10^{-7}\times10^3} = e^{-7\times10^{-4}} \approx 0.9993.$$

The proportion decayed is

$$1 - \frac{M}{M_0} \approx 1 - 0.9993 = 0.0007.$$

3. How long before 95% has decayed?

$$\frac{M}{M_0} = 1 - 0.95 = 0.05 = e^{-7\times 10^{-7}t},$$

taking logarithms of both sides we have

$$\ln(0.05) = -7 \times 10^{-7} t$$

which implies

$$t = \frac{\ln(0.05)}{-7 \times 10^{-7}} \approx \frac{-2.996}{-7 \times 10^{-7}} \approx 4.3 \times 10^{6} \text{ (years)}.$$

WARNING: Half-life $T_{\frac{1}{2}}$ of a particular element does not mean that in $2 \times T_{\frac{1}{2}}$, the element will completely decay.

Population growth

Example
Suppose a certain bacterium divides each hour. Each hour the population doubles:

Hours	1	2	3	4	5	...
Population	2	4	8	16	32	...

After t hours you have 2^t times more bacteria than what you started with. In general, we write it as an exponential form.

If a population, initially P_0 grows exponentially with growth rate λ (where $\lambda > 0$), then at time t, the population is

$$P(t) = P_0 e^{\lambda t}.$$

Example
Bacterium divides every hour.

1. What is the growth rate?

   We know that when $t = 1$ hour, we are supposed to have

   $$P = 2P_0,$$

   so

   $$2P_0 = P_0 e^{\lambda \cdot 1} \quad \Longrightarrow \quad 2 = e^{\lambda} \quad \Longrightarrow \quad \lambda = \ln 2 \approx 0.693.$$

2. How long for 1 bacterium to become 1 billion?

   $$P_0 = 1, \quad \lambda = 0.693, \quad P = 10^9,$$

   therefore we may write

   $$P = P_0 e^{\lambda t} \quad \Longrightarrow \quad 10^9 = e^{0.693 t},$$

taking logarithms of both sides and re-arranging for t, we have

$$t = \frac{9 \ln 10}{0.693} \approx 30 \text{ hours}.$$

Interest rate

An annual interest rate of 5% tells you that £100 investment at the start of the year grows to £105. Each subsequent year you leave your investment, it will be multiplied by the factor 1.05.

In general, if you initially invest M_0 (amount) with an annual interest rate r (given as percentage/100), then after t years you have

$$A = M_0(1 + r)^t,$$

where A is the future value. We could write this as an exponential as follows:

$$A(t) = M_0 e^{\lambda t} = M_0(1 + r)^t.$$

Taking logarithms we have

$$\lambda t = t \ln(1 + r),$$

so we may write

$$A(t) = M_0 e^{\ln(1+r)t}.$$

Chapter 3

Integration

3.1 The basic idea

If we have a distance-time graph, the gradient of the graph gives us the velocity at that point. In the previous chapter, we learnt how to find the gradient at a point on a curve.

If we have a velocity-time graph, the area under the curve gives us the distance travelled. In this chapter, we will learn how to find this.

The method of finding the area under a curve is called **integration**. We will write the area under a curve $y = f(x)$ between $x = a$ and $x = b$ as

$$\int_a^b f(x)\ dx.$$

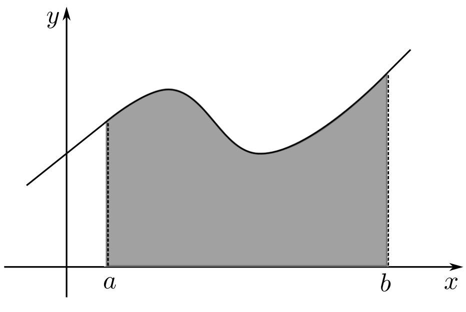


Figure 3.1: $\int_a^b f(x)\ dx$.

3.1.1 Finding the area under a curve

To find the area under a curve, we begin in a similar way to how we began differentiation:

1. We divide the interval $a \leq x \leq b$ into small pieces, each of length h.
2. We build a rectangle on each piece, where the top touches the curve.

3. We calculate the total area of the rectangles.

As we make h get smaller and smaller, the area of the rectangles gets closer and closer to the area under the curve.

Example
Consider the function $f(x) = x$ on the interval $0 \le x \le 1$.
We divide $[0, 1]$ into n equal pieces, each of width $h = \frac{1}{n}$. The divisions occur at

$$0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{k-1}{n}, \frac{k}{n}, \dots, \frac{n-1}{n}, 1$$

or

$$0, h, 2h, \dots, n-1)h, 1$$

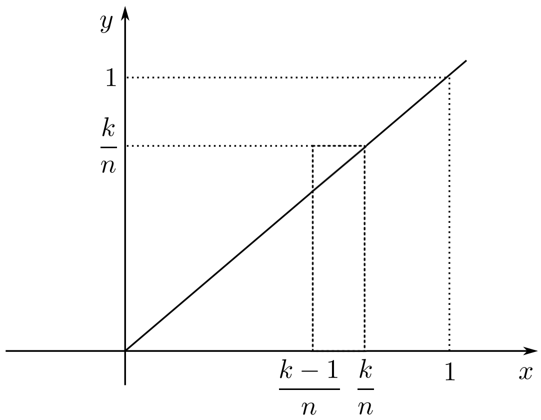


Figure 3.2: *The rectangle between $(k-1)h$ and kh.*

The rectangle between $(k-1)h$ and kh will have height $f(kh) = kh$, and the area of this rectangle is

$$\underbrace{kh}_{\text{height}} \cdot \underbrace{h}_{\text{width}} = kh^2.$$

The sum of the area of all rectangles on the interval is

$$\begin{aligned} h^2 + 2h^2 + \cdots + nh^2 &= h^2(1 + 2 + \cdots + n) \\ &= h^2 \frac{n(n+1)}{2} \\ &= h^2 \frac{\frac{1}{h}(\frac{1}{h} + 1)}{2} \\ &= \frac{1+h}{2}. \end{aligned}$$

As $h \to 0$, $\frac{1+h}{2} \to \frac{1}{2}$. Therefore,

$$\int_0^1 x \, dx = \frac{1}{2}.$$

As we did with differentiation, we would like to find faster methods of finding the area under a curve. To do this, we relate integration and differentiation.

3.1.2 The fundamental theorem of calculus

It is often treated as obvious that integration and differentiation are opposites. However, it is un-obvious enough that mathematicians have a big theorem about it:

Theorem: Fundamental Theorem of Calculus

$$\int_a^b g'(x)\ dx = g(b) - g(a)$$

In other words, if we are trying to find

$$\int_a^b f(x)\ dx$$

then if we can find a function $F(x)$ so that $F'(x) = f(x)$,

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Definition
We call $F(x)$ the **antiderivative** of $f(x)$.

The aim of this chapter is to learn methods for finding the antiderivative.

3.1.3 Indefinite and definite integrals

Let $f(x)$ be a function. If $F(x)$ is the antiderivative of $f(x)$, then $F(x) + 4$ is also the antiderivative of $f(x)$.

Proof:

$$\begin{aligned}\frac{d}{dx}(F(x)+4) &= \frac{d}{dx}(F(x)) + \frac{d}{dx}(4)\\ &= f(x) + 0\end{aligned}$$

□

Similarly, $F(x) + c$ will be the antiderivative of $f(x)$ for any $c \in \mathbb{R}$. When finding an integral without limits, we must include this constant term.

Definition
The **indefinite integral** of a function $f(x)$ is

$$\int f(x)\ dx = F(x) + c$$

where $F(x)$ is any derivative of $f(x)$. Ususally, we pick $F(x)$ as the antiderivative without a constant term.

Definition

$$\int_a^b f(x)\ dx = F(x)$$

is called the **definite integral**.

3.2 Finding integrals

3.2.1 Polymonials and other powers

To find indefinite integrals, we are going to look for functions which will have the correct derivative.

$$\int ax^b\ dx = \frac{ax^{b+1}}{b+1} + c$$

Proof:

$$\frac{d}{dx}\left(ax^B\right) = aBx^{B-1}$$

so

$$\frac{d}{dx}\left(\frac{ax^B}{B}\right) = ax^{B-1}.$$

Replacing B with $b+1$ gives the correct result.

□

Example

$$\int x^2\ dx = \frac{x^3}{3} + c$$

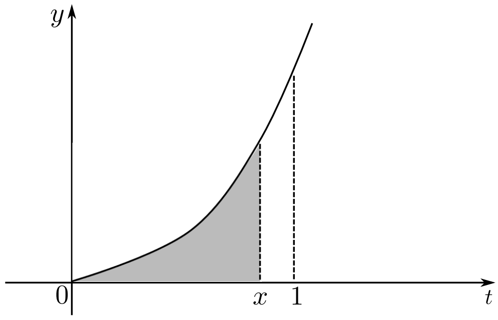
Example

$$\int x^2 + x^3 \; dx = \frac{x^3}{3} + \frac{x^4}{4} + c$$

Example
To find

$$\int_0^x t^2 \; dt,$$

or the area under $y = t^2$ between $t = 0$ and $t = x$:



We first find

$$\int t^2 \; dt = \frac{t^3}{3} + c.$$

Then

$$\begin{aligned}\int_0^x t^2 \; dt &= \frac{x^3}{3} + c - \frac{0^3}{3} - c \\ &= \frac{x^3}{3}.\end{aligned}$$

For $x = 1$, the area under $y = t^2$ between $t = 0$ and $t = 1$ is

$$\begin{aligned}\int_0^1 t^2 \; dt &= \frac{1^3}{3} \\ &= \frac{1}{3}.\end{aligned}$$

When finding definite integrals, the constants will always cancel, so can be ignored. We often write the working like this, with the indefinite integral in brackets:

Example

$$\begin{aligned}\int_1^3 3x^2 \; dx &= \left[x^3\right]_1^3 \\ &= 3^3 - 1^3 \\ &= 26\end{aligned}$$

This also works with negative and fractional powers.

Example
Suppose we want to integrate the function $1/x^2$ over the interval $[1, 2]$. That is, we want to calculate

$$\int_1^2 \frac{1}{x^2}\, dx.$$

If we put $F(x) = -1/x$, then

$$F'(x) = \frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}.$$

So we can write

$$\int_1^2 \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^2 = \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right) = \frac{1}{2}.$$

The integral $\int_1^2 \frac{1}{x^2}\, dx$ represents the area under the curve $y = \frac{1}{x^2}$ between 1 and 2, therefore we understand that this integral makes some geometrical sense.

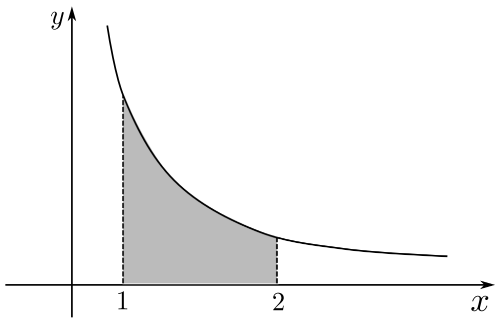


Figure 3.3: Integrating to find the shaded area under the curve $y = \frac{1}{x^2}$ on the interval $[1, 2]$.

Example

$$\begin{aligned} \int \sqrt{x}\, dx &= \int x^{\frac{1}{2}}\, dx \\ &= 2x^{\frac{3}{2}} + c \end{aligned}$$

x^{-1} is a special case, as using the same rule would require division by 0:

$$\int \frac{1}{x}\, dx = \ln|x| + c$$

Proof: This is true because for $x > 0$

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

and for $x < 0$,

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}.$$

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

□

3.2.2 Exponential functions

$$\int e^x \; dx = e^x + c$$

Proof: This is true because

$$\frac{d}{dx}e^x = e^x.$$

□

$$\int a^x \; dx = \frac{a^x}{\ln a} + c$$

Proof: This is true because

$$\begin{aligned}\frac{d}{dx}\left(\frac{a^x}{\ln a}\right) &= \frac{1}{\ln a}\frac{d}{dx}\left(e^{x\ln a}\right)\\ &= \frac{1}{\ln a}e^{x\ln a}\ln a\\ &= e^{x\ln a}.\end{aligned}$$

□

3.2.3 Trigonometric functions

$$\int \cos x \; dx = \sin x + c, \qquad \text{since } \frac{d}{dx}(\sin x) = \cos x,$$

$$\int \sin x \; dx = -\cos x + c, \qquad \text{since } \frac{d}{dx}(-\cos x) = \sin x.$$

3.3 Rules for integration

Instead of making a longer and longer list of functions and their antiderivatives, we are going to learn some rules for integration and use these to work out harder integrals

3.3.1 Sum rule and constants

Sum Rule

$$\int (f(x) + g(x)) \; dx = \int f(x) \; dx + \int g(x) \; dx \tag{3.1}$$

Multiplication by a constant

$$\int K f(x) \; dx = K \int f(x) \; dx \tag{3.2}$$

Both these rules follow from the equivalent rules differentiation.

3.3.2 A special case

Let us consider the derivative of the logarithm of some general function $f(x)$,:

$$\begin{aligned} \frac{d}{dx}(\ln(f(x))) &= \frac{1}{f(x)} \cdot \frac{d}{dx}(f(x)) \\ &= \frac{f'(x)}{f(x)} \end{aligned}$$

This implies that:

$$\int \frac{f'(x)}{f(x)} \; dx = \ln(f(x)) + c$$

Example
Consider the the following integral:

$$I = \int \frac{2x+5}{x^2+5x+3}\,dx.$$

Now, if we choose $f(x) = x^2 + 5x + 3$, then $f'(x) = 2x + 5$. So, if we differentiate $\ln(f(x))$, in this case we have

$$\frac{d}{dx}\left[\ln(x^2+5x+3)\right] = \frac{2x+5}{x^2+5x+3},$$

by the chain rule. Thus, we know the integral must be

$$I = \ln(x^2+5x+3) + c.$$

Example
Consider the following integral:

$$I = \int \frac{3}{2x+2}\,dx.$$

Now, if we choose $f(x) = 2x + 2$ then $f'(x) = 2$. However, the numerator of the integrand is 3. Not to worry, as we can simply re-write or manipulate the initial integral as follows:

$$I = \int \frac{3}{2x+2}\,dx = 3\int \frac{1}{2}\frac{2}{2x+2}\,dx = \frac{3}{2}\int \frac{2}{2x+2}\,dx.$$

Since 3/2 is a constant, which we are able to take out of the integral sign, we need not worry about this and can proceed with the integration using what we have learnt above, giving

$$I = \frac{3}{2}\ln(2x+2) + c.$$

To check, we differentiate the above expression, so

$$\frac{dI}{dx} = \frac{d}{dx}\left[\frac{3}{2}\ln(2x+2) + c\right] = \frac{3}{2}\cdot\frac{1}{2x+2}\cdot 2,$$

which is correct!

This "special case" is an example of a method called *substitution*, and is not limited to integrals which give you logarithms.

3.3.3 Substitution

We can use substitution to convert a complicated integral into a simple one.

Example
We want to find

$$\int (2x+3)^{100}\ dx.$$

We make the substitution

$$u = 2x + 3.$$

This means that

$$\frac{du}{dx} = 2$$

and so

$$dx = \frac{1}{2}du.$$

So we calculate the integral as follows:

$$\begin{aligned}\int (2x+3)^{100}\ dx &= \int u^{100}\cdot\frac{1}{2}\ du \\ &= \frac{1}{2}\int u^{100}\ du \\ &= \frac{1}{2}\cdot\frac{1}{101}u^{101} \\ &= \frac{1}{202}(2x+3)^{101} + c.\end{aligned}$$

We can check the result by performing the following differentiation:

$$\frac{d}{dx}\left[\frac{1}{202}(2x+3)^{101} + c\right] = \frac{101}{202}(2x+3)^{100}\cdot 2 = (2x+3)^{100},$$

which is correct.

$\frac{dx}{du}$ is not really a fraction so cannot really be split up as we did here. What we are actually doing is using the following:

Integration by Substitution

$$\int_a^b f(u(x))\ dx = \int_{u(a)}^{u(b)} f(u)\frac{dx}{du}\ du.$$

Substituting the u and $\frac{dx}{du}$ into this formula is equivalent to the splitting up of $\frac{dx}{du}$ which we did. The splitting method gives the correct answer and can be thought of as an easier way to remember the method we are actually using.

Example

$$\int x(x+1)^{50}\ dx$$

Try the substitution $u = x + 1$ (i.e. $x = u - 1$). This gives:

$$\frac{du}{dx} = 1$$

$$dx = du$$

So we have

$$\begin{aligned}
\int x(x+1)^{50}\ dx &= \int (u-1)u^{50}\ du \\
&= \int u^{51}\ du - \int u^{50}\ du \\
&= \frac{1}{52}u^{52} - \frac{1}{51}u^{51} + c \\
&= \frac{1}{52}(x+1)^{52} - \frac{1}{51}(x+1)^{51} + c
\end{aligned}$$

Check:

$$\frac{d}{dx}\left(\frac{1}{52}(x+1)^{52} - \frac{1}{51}(x+1)^{51} + c\right) = (x+1)^{51} - (x+1)^{50} = (x+1)^{50} \cdot x,$$

which is correct.

Example

$$\int \frac{1}{x \ln x}\ dx$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$dx = x\ du$$

$$\begin{aligned}
\int \frac{1}{x \ln x}\ dx &= \int \frac{1}{xu} \cdot x\ du \\
&= \int \frac{1}{u}\ du \\
&= \ln|u| + c \\
&= \ln|\ln x| + c.
\end{aligned}$$

Check:

$$\frac{d}{dx}\left(\ln|\ln x|\right) = \frac{1}{\ln x} \cdot \frac{1}{x},$$

which is correct.

Example

$$\int \frac{1}{1+\sqrt{x}}\ dx$$

Let $u = 1 + \sqrt{x}$.

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$dx = 2x^{\frac{1}{2}}\ du = 2(u-1)\ du$$

$$\begin{aligned}
\int \frac{1}{1+\sqrt{x}}\ dx &= \int \frac{1}{u} \cdot 2(u-1)\ du \\
&= 2\int \left(1 - \frac{1}{u}\right)\ du \\
&= 2\int\ du - 2\int \frac{1}{u}\ du \\
&= 2u - 2\ln|u| + c \\
&= 2(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + c.
\end{aligned}$$

Check:

$$\frac{d}{dx}\left(2(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + c\right) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(1+\sqrt{x})} = \frac{1+\sqrt{x}-1}{\sqrt{x}(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}}.$$

Example

$$\int \sin(3x+1)\ dx$$

Let $u = 3x + 1$.

$$\frac{du}{dx} = 3$$

$$dx = \frac{1}{3}[u]$$

$$\begin{aligned}
\int \sin 3x + 1\ dx &= \frac{1}{3}\int \sin u\ du \\
&= -\frac{1}{3}\cos u + c \\
&= -\frac{1}{3}\cos(3x+1) + c.
\end{aligned}$$

Check:

$$\frac{d}{dx}\left(-\frac{1}{3}\cos(3x+1) + c\right) = +\frac{1}{3} \cdot 3 \cdot \sin(3x+1) = \sin(3x+1).$$

Example

$$\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\ dx$$

Let $u = \frac{1}{x}$.

$$\frac{du}{dx} = -\frac{1}{x^2}$$
$$dx = -x^2\ du$$

$$\begin{aligned}\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\ dx &= \int \frac{\sin u}{x^2} \cdot (-x^2)\ du \\ &= -\int \sin u\ du \\ &= \cos u + c \\ &= \cos\left(\frac{1}{x}\right) + c.\end{aligned}$$

Check:

$$\frac{d}{dx}\left(\cos\left(\frac{1}{x}\right) + c\right) = -\sin\left(\frac{1}{x}\right) \cdot (-x^{-2}) = \frac{\sin\left(\frac{1}{x}\right)}{x^2},$$

which is correct.

3.3.4 Trigonometric substitution

Example

We know that

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c,$$

since

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Actually, we can work out this integral by a substitution like $x = \sin u$ because we know that

$$1 - \sin^2 u = \cos^2 u,$$

and

$$\frac{dx}{du} = \cos u \quad \text{or} \quad dx = \cos u\ du.$$

Thus, we calculate the integral as

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\ du = \int\ du = u + c = \sin^{-1} x + c.$$

Example

$$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + c, \quad \text{since} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Let us try the following

$$x = \tan\theta$$

$$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

$$dx = (1 + \tan^2\theta)\ d\theta$$

$$\int \frac{1}{1+x^2}\ dx = \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta)\ d\theta = \int\ d\theta = \theta + c = \tan^{-1} x + c.$$

Example

$$\int \frac{1}{1+2x^2}\ dx$$

This is similar to the previous example. If we try

$$\sqrt{2}x = \tan\theta$$

$$dx = \frac{1}{\sqrt{2}}(1+\tan^2\theta)\ d\theta$$

$$\theta = \tan^{-1}(\sqrt{2}x)$$

$$\begin{aligned}
\int \frac{1}{1+2x^2}\ dx &= \int \frac{1}{1+(\sqrt{2}x)^2}\ dx \\
&= \frac{1}{\sqrt{2}} \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta)\ d\theta \\
&= \frac{1}{\sqrt{2}}\theta + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}x) + c.
\end{aligned}$$

3.3.5 Integration by parts

This is equivalent to the product rule for integration. Suppose we have two function $u(x)$ and $v(x)$. Then the product rule states

$$\frac{d}{dx}(uv) = u'v + uv'.$$

Rearranging the above gives

$$uv' = \frac{d}{dx}(uv) - u'v.$$

Integrating both sides we get

$$\begin{aligned}
\int uv'\ dx &= \int \frac{d}{dx}(uv)\ dx - \int u'v\ dx \\
&= uv - \int u'v\ dx.
\end{aligned}$$

So we write the rule for integration by parts as:

Integration by Parts

$$\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx.$$

Example

$$\int xe^x\ dx$$

Let:

$$u = x, \quad v' = e^x$$

This means that:

$$u' = 1, \quad v = e^x.$$

Therefore, we can calculate the integral as follows:

$$\begin{aligned}\int xe^x\ dx &= \int uv'\ dx \\ &= uv - \int u'v\ dx \\ &= xe^x - \int e^x\ dx \\ &= xe^x - e^x + c.\end{aligned}$$

Check:

$$\frac{d}{dx}\left[xe^x - e^x + c\right] = e^x + xe^x - e^x = xe^x.$$

Example

$$\int \ln x\ dx$$

This is the same as

$$\int 1 \cdot \ln x\ dx$$

We choose:

$$u = \ln x, \quad v' = 1$$

This means that:

$$u' = \frac{1}{x}, \quad v = x.$$

So we calculate the integral as

$$\begin{aligned}\int \ln x \; dx &= \int 1 \cdot \ln x \; dx \\ &= \int uv' \; dx \\ &= uv - \int u'v \; dx \\ &= x \ln x - \int \frac{1}{x} \cdot x \; dx \\ &= x \ln x - x + c.\end{aligned}$$

Check:

$$\frac{d}{dx}\left[x \ln x - x + c\right] = \ln x + x \cdot \frac{1}{x} - 1 = \ln x.$$

Example

$$\int e^x \cos x \; dx$$

Let:

$$u = \cos x, \quad v' = e^x$$

This means that

$$u' = -\sin x, \quad v = e^x$$

So we write our integral as

$$\begin{aligned}\int e^x \cos x \; dx &= \int uv' \; dx \\ &= uv - \int u'v \; dx \\ &= e^x \cos x + \int e^x \sin x \; dx.\end{aligned}$$

Now, we have an integral similar to what we started with, so let us integrate this by parts too, choosing

$$\bar{u} = \sin x, \quad \bar{v}' = e^x \qquad \bar{u}' = \cos x, \quad \bar{v} = e^x.$$

So our original integral becomes

$$\begin{aligned}\int e^x \cos x \; dx &= e^x \cos x + \int \bar{u}\bar{v}' \; dx \\ &= e^x \cos x + \bar{u}\bar{v} - \int \bar{u}'\bar{v} \; dx \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x \; dx.\end{aligned}$$

Note, now on the RHS we have the same integral we started with. Rearranging this, we can make the integral the subject:

$$\int e^x \cos x \; dx = e^x \cos x + e^x \sin x - \int e^x \cos x \; dx,$$

$$\therefore \quad 2\int e^x \cos x \; dx = e^x \cos x + e^x \sin x.$$

So finally, we can write:

$$\int e^x \cos x \; dx = \frac{1}{2}\left[e^x \left(\cos x + \sin x\right)\right] + c,$$

Check:

$$\frac{d}{dx}\left[\frac{1}{2}\left[e^x \left(\cos x + \sin x\right)\right] + c\right] = \frac{1}{2}\left\{e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)\right\} = e^x \cos x$$

3.3.6 Partial fractions

We can sometimes split polynomial fractions into smaller parts.

Example

$$\frac{1}{x^2-1}$$

First factorise the denominator:

$$x^2 - 1 = (x+1)(x-1)$$

Now write:

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1},$$

where A and B are constants to be found. Multiplying everything by $(x-1)(x+1)$ gives:

$$1 = A(x+1) + B(x-1)$$

Substituting in $x = 1$ gives

$$1 = 2A.$$

Substituting in $x = -1$ gives

$$1 = -2B.$$

Therefore,

$$\begin{aligned} A &= \frac{1}{2}, \\ B &= -\frac{1}{2}. \end{aligned}$$

Hence:

$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Definition
Splitting the fraction as above is called splitting into **partial fractions**.

We can use partial fractions for integration:

Example

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx\\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c\\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c.\end{aligned}$$

Example

Consider the integral

$$\int \frac{x^2+6x+1}{3x^2+5x-2}\,dx.$$

To split into partial fractions, the numerator needs to be of a lower degree than the denominator. So we manipulate as follows:

$$\begin{aligned}\frac{x^2+6x+1}{3x^2+5x-2} &= \frac{\frac{1}{3}(3x^2+18x+3)}{3x^2+5x-2}\\ &= \frac{\frac{1}{3}(3x^2+5x-2+13x+5)}{3x^2+5x-2}\\ &= \frac{1}{3}\left[1+\frac{13x+5}{3x^2+5x-2}\right].\end{aligned}$$

Now we factorise the denominator:

$$3x^2+5x-2=(3x-1)(x+2)$$

Next we write split the fraction into partial fractions

$$\frac{13x+5}{3x^2+5x-2} = \frac{A}{3x-1} + \frac{B}{x+2} = \frac{(A+3B)x+2A-B}{(3x-1)(x+2)}.$$

Matching the coefficients on the numerator we must have

$$\begin{aligned}A+3B&=13\\ 2A-B&=5\end{aligned}$$

Solving simultaeneously we have the solution $A=4$ and $B=3$. So the integral becomes

$$\begin{aligned}\int \frac{x^2+6x+1}{3x^2+5x-2}\,dx &= \frac{1}{2}\int 1+\frac{13x+5}{3x^2+5x-2}\,dx\\ &= \frac{1}{3}\int\,dx + \frac{1}{3}\int \frac{4}{3x-1}\,dx + \frac{1}{3}\int \frac{3}{x+2}\,dx\\ &= \frac{1}{3}x + \frac{4}{9}\ln|3x-1| + \ln|x+2| + c.\end{aligned}$$

3.4 Some difficulties

Example
Consider the integral of $1/x^2$ on the interval $[-1, 1]$.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = \left[-\frac{1}{1}\right] - \left[-\frac{1}{-1}\right] = -2.$$

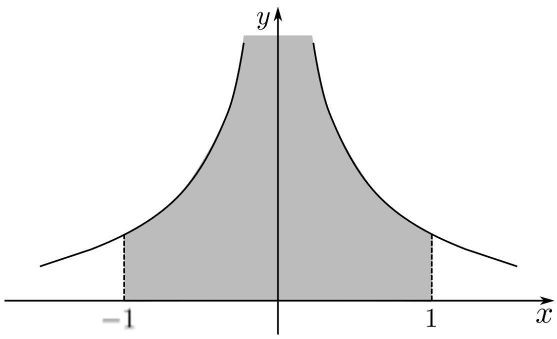


Figure 3.4: Integrating to find the shaded area under the curve $y = \frac{1}{x^2}$ on the interval $[-1, 1]$?

However, the area under the curve in this interval is not -2! What is wrong here?

In the above example, the integral was not the area under the curve because there was an asymptote at $x = 0$. We must be sure that the function is defined over the whole domain before we integrate.

Example
Consider the function $f(x) = x^3$. Then the integral over the interval $[-1, 1]$ is

$$\int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0.$$

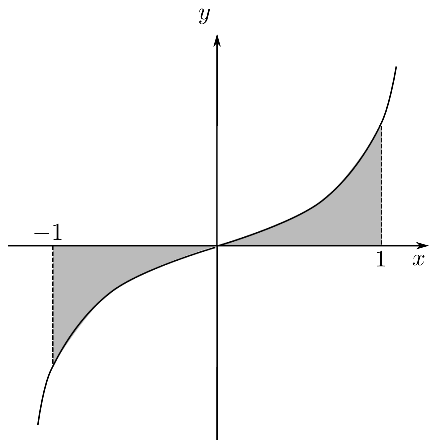


Figure 3.5: The area under the curve $y = x^3$ on the interval $[-1, 1]$.

In this case, the areas cancel out as area below the x-axis is negative. The shaded area is actually given by

$$\int_0^1 x^3\,dx + \left|\int_{-1}^0 x^3\,dx\right| = \left[\frac{1}{4}x^4\right]_0^1 + \left|\left[\frac{1}{4}x^4\right]_{-1}^0\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

3.5 Applications of integration

There are many real life situations in which quantities are derivatives or integrals of each other. Some are given in the table below. Differentiating moves to the right in the table (following the arrows). Integrating moves to the left (against the arrows).

Distance	→	Speed	→	Acceleration
Energy	→	Force		
Prices	→	Inflation	→	Rate of change of inflation
Debt	→	Deficit	→	Increase/reduction in deficit
Bank balance	→	Interest		
Population size	→	Birth and death rates		
Something	→	Rate of change of that thing		

All of the above would involve $\frac{d}{dt}$ as they are the rate of change of a quantity over **time**.

3.5.1 Finding a distance by the integral of velocity

If you know the velocity $v(t)$, then the distance s as a function of time, i.e. $s = s(t)$, is

$$s(t) = \int v(t)\,dt, \quad (\text{since } s'(t) = v(t)).$$

Similarly, if you know the acceleration $a(t)$, then the velocity can be found by the integral of $a(t)$,

$$v(t) = \int a(t)\,dt \quad (\text{since } v'(t) = a(t))$$

Example
A ball is thrown down from a tall building with an initial velocity of 100ft/sec. Then its velocity after t seconds is given by $v(t) = 32t + 100$. How far does the ball fall between 1 and 3 seconds of elapsed time?
First let us write the distance as the integral of the velocity, that is

$$s(t) = \int v(t) \ = \int 32t + 100\,dt = 16t^2 + 100t + C.$$

Then the distance fallen is given by

$$s(3) - s(1) = (16t^2 + 100t + C)\big|_{t=3} - (16t^2 + 100t + C)\big|_{t=1} = 328 \text{ ft}.$$

Notice that

$$\begin{aligned} (16t^2 + 100t + C)\big|_{t=3} - (16t^2 + 100t + C)\big|_{t=1} &= (16t^2 + 100t + C)\big|_1^3 \\ &= \int_1^3 (32t + 100)\,dt \\ &= \int_1^3 v(t)\,dt. \end{aligned}$$

3.5.2 Finding the area between two curves

Example
To find the area between the curves $y = x^2$ and $y = 50 - x^2$...

3.6 Numerical integration

Consider evaluating the definite integral

$$\int_a^b f(x)\,dx.$$

For the vast majority of function, the antiderivataive is not known. For example, we can't find:

$$\int \sqrt{1 + x^3}\,dx \quad \text{or} \quad \int e^{x^2}\,dx.$$

When applying integration to a real application this is a problem.

When an impossible integral is encountered, we must use a **numerical method** to approximate the answer.

3.6.1 Trapezium method

We want to estimate the integral of $f(x)$ on the interval $[a, b]$, which represents the area under the curve $y = f(x)$ from a to b.

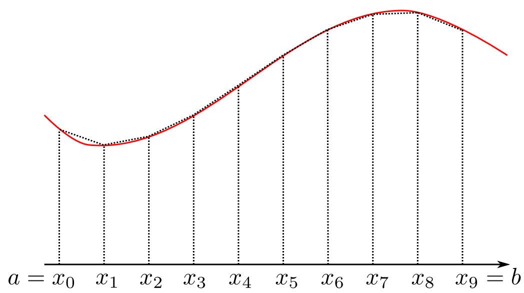


Figure 3.6: Forming trapeziums with height of the sides dictated by the curve $y = f(x)$ over the interval $[a, b]$.

We choose n number of pieces. Divide the interval $a \le x \le b$ into n (equal) pieces with points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

On each piece of the interval, we build a trapezium by joining points on the curve by a straight line. We calculate the total area by summing all the area of the trapezia. This is our estimate of the integral.

To start with, let h be the width of one piece of the interval, i.e.

$$h = \frac{b-a}{n},$$

then we have

$$x_k = x_0 + kh, \quad k = 0, 1, 2, \ldots, n. \quad x_0 = a, \quad x_n = b.$$

Let us consider the trapezium based on the piece $[x_{k-1}, x_k]$, whose width is h. The height of the sides of the trapezium are $f(x_{k-1})$ and $f(x_k)$. So the area is

$$h\frac{f(x_{k-1}) + f(x_k)}{2}.$$

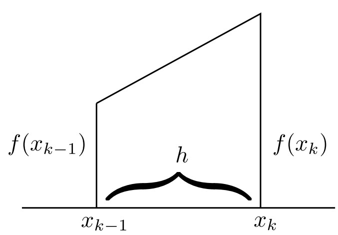


Figure 3.7: *Trapezium constructed over each piece of the interval, where each piece has width h.*

Then the total area under the curve over $[a, b]$ is the sum:

$$\begin{aligned}
\text{Area} &= h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \cdots + h\frac{f(x_{n-1})+f(x_n)}{2} \\
&= \frac{h}{2}\left[(f(x_0)+f(x_1)) + (f(x_1)+f(x_2)) + \cdots + (f(x_{n-1})+f(x_n))\right] \\
&= \frac{h}{2}\left[f(x_0) + 2(f(x_1)+f(x_2)+\cdots+f(x_{n-1})) + f(x_n)\right] \\
&= \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right].
\end{aligned}$$

We can think of the sum as follows, we have the two outer sides of the first and last trapezium, then every trapezium in-between shares its sides with its neighbour, therefore we require two lots of the interior sides.

Example
Using the trapezium method, estimate

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

We choose $n = 4$, then

$$h = \frac{1-0}{4} = \frac{1}{4}, \quad x_k = kh, \quad k = 0, 1, 2, 3, 4.$$

Also, note that

$$f(x) = \frac{1}{1+x^4}, \quad \text{i.e.} \quad f(x_k) = \frac{1}{1+x_k^4}.$$

Therefore, we have

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^4}\,dx &\approx \frac{h}{2}\left[f(x_0) + 2(f(x_1)+f(x_2)+f(x_3)) + f(x_4)\right] \\
&= \frac{1}{8}\left[f(0) + 2(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)) + f(1)\right] \\
&= \frac{1}{8}\left[1 + 2\left(\frac{256}{257} + \frac{16}{17} + \frac{256}{337}\right) + \frac{1}{2}\right] \\
&= 0.862.
\end{aligned}$$

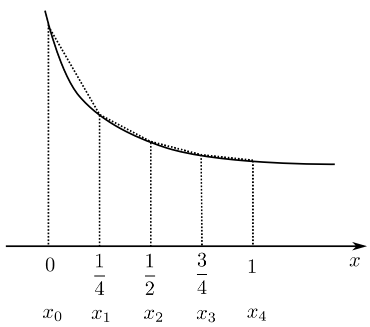


Figure 3.8: Numerically integrating under $y = 1/(1+x^4)$. Dividing interval into 4 pieces of width $h = 1/4$.

This is an over-estimate of the integral since $y = f(x)$ is convex (i.e. it curves up like a cup). If it were concave (i.e. curved down like a cap), then you would have an under-estimate.

Chapter 4

Differential Equations

In many applications, we have equations relating a functions and its derivatives. For example:

> In radioactive decay, we know $\frac{dy}{dt} = \lambda y$, where y is the number of particles of radioactive material.
>
> Inflation is expressed as a percentage of current prices, so $\frac{dp}{dt} = ip$, where p is prices and i is inflation.
>
> The movement of an object on a spring follows the equation $\frac{d^2y}{dx^2} = -\omega y$.

Equations like these are called **(ordinary) differential equations** or **ODEs**.

In this chapter we will look at methods for solving ODEs.

4.1 Terminology

Definition
An equation involving y and $\frac{dy}{dx}$ is called a **first order** ODE.
An equation involving y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ is called a **second order** ODE.
When solving ODEs, solutions involving constants often appear. These are called **general solutions** of ODEs.

Extra information is often given to give the constants in the general solution a value.

Definition
The extra information given is called the **boundary conditions**.
A problem with an ODE and boundary conditions is called an **initial value prob-**

lem or **IVP**.

Definition
An n-th order differential equation is linear if it can be written in the form:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_{n-2}(x)y^{(n-2)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

or

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x),$$

where a_i $(i = 0, 1, 2, \ldots, n)$ and f are known functions of x.

Example
$y' + 2y = e^x$ is a first-order linear differential equation.
$yy' = x$ is a first-order non-linear differential equation.
$y'' - e^x y' + y = x$ is a second-order linear differential equation.

Definition
If $f(x) \equiv 0$, then the differential equation is said to be **homogeneous**; otherwise, we say the equation is **non-homogenous** or **inhomogenous**.

Example
$y' + 2y = e^x$ is a non-homogeneous differential equation.
$y' + 2y = 0$ is a homogeneous differential equation.

4.2 First order differential equations

Here we will consider different techniques to solve first order ODEs.

4.2.1 Separation of variables

First order ODEs can be written in the form

$$\frac{dy}{dx} = f(x, y).$$

For example

$$y' = -2xy + e^x,$$

$$\tfrac{dy}{dx} = \pm\sqrt{x^3 - 2\ln y + 4e^x},$$

$$y' = x/y^2.$$

Definition
A function $f(x, y)$ is **separable** if it can be written as

$$f(x, y) = g(x)h(y).$$

Example
Let $f(x, y) = \frac{x}{y^2} = x \cdot \frac{1}{y^2}$.
This is seperable because

$$\frac{x}{y^2} = x \cdot \frac{1}{y^2}$$

so $f(x, y) = g(x)h(y)$ where

$$g(x) = x$$
$$h(y) = \frac{1}{y^2}$$

When f is seperable, we can solve $\frac{dy}{dx} = f(x, y)$ by a method called **separating the variables**.

Example
Consider the differential equation

$$y' = \lambda y.$$

We already know the solution to this equation. Now let us see how to derive it using separation of variables.

$$\frac{dy}{dt} = \lambda y,$$

taking all things relating to y to the left, and for t to the right, we have

$$\frac{1}{y} dy = \lambda dt.$$

Integrating both sides we have

$$\int \frac{1}{y}\ dy = \int \lambda\ dt,$$

hence, using what we have learnt in previous chapters we get

$$\ln y = \lambda t + C.$$

Finally, re-arranging for y, we have

$$y = e^{\lambda t + C} = Ae^{\lambda t}, \quad A = e^C.$$

Example
Consider the equation

$$\frac{dy}{dx} = xy,$$

following the procedure as in the previous example, we have

$$\frac{1}{y}\ dy = x\ dx.$$

Integrating both sides we have

$$\begin{aligned} \int \frac{1}{y}\ dy &= \int x\ dx, \\ \implies \qquad \ln y &= \frac{1}{2}x^2 + C. \end{aligned}$$

Taking exponentials of both sides in order to re-arrange for y, we get

$$y = e^{\frac{1}{2}x^2 + C} = Ae^{\frac{1}{2}x^2}, \quad A = e^C.$$

We can check if this satisfies the original equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(Ae^{\frac{1}{2}x^2}\right) = Ae^{\frac{1}{2}x^2 +} \cdot \frac{1}{2} \cdot 2x = xy.$$

Example
Consider the differential equation

$$y^2 y' = x.$$

We first write it in the form $y' = f(x, y)$, i.e.

$$\frac{dy}{dx} = \frac{x}{y^2},$$

now we realises that we can apply separation of variable, so

$$y^2 dy = x\ dx,$$

$$\begin{aligned} \implies \qquad \int y^2 dy &= \int x\ dx \\ \implies \qquad \frac{1}{3}y^3 &= \frac{1}{2}x^2 + C, \\ \implies \qquad y &= \left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}, \end{aligned}$$

where C' is some constant (different to C, since we multiplied through by 3). Again, we check the solution satisfies the equation

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right] \\ &= \frac{1}{3}\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \cdot \frac{3}{2} \cdot 2x \\ &= x\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \\ &= x\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right]^{-2} \\ &= \frac{x}{y^2}. \end{aligned}$$

Example
Consider the following initial-value problem:

$$\frac{dy}{dx} = y^2(1 + x^2), \quad y(0) = 1.$$

First, we find the general solution, note, we can use separation of variables in this example, so

$$\frac{1}{y^2}\ dy = (1 + x^2)\ dx$$

$$\begin{aligned} \implies &\quad \int \frac{1}{y^2}\ dy = \int (1 + x^2)\ dx \\ \implies &\quad -\frac{1}{y} = x + \frac{1}{3}x^3 + C \\ \implies &\quad y = -\frac{1}{x + \frac{1}{3}x^3 + C}. \end{aligned}$$

Now check that the general solution satisfies the original differential equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{1}{x + \frac{1}{3}x^3 + C}\right) = \frac{1 + x^2}{\left(x + \frac{1}{3}x^3 + C\right)^2} = (1 + x^2)y^2.$$

Now it remains to find the constant C, by applying the condition $y(0) = 1$, i.e. we put $x = 0$.

$$y(0) = -\frac{1}{0 + \frac{1}{3} \cdot 0^3 + C} = -\frac{1}{C} = 1, \quad \implies \quad C = -1.$$

So the solution to the initial value problem is

$$y = \frac{1}{1 - x - \frac{1}{3}x^3}.$$

Example
Consider the initial value problem

$$e^y y' = 3x^2, \quad y(0) = 2.$$

First, find the general solution,

$$\begin{aligned} & e^y y' = 3x^2 \\ \implies & \frac{dy}{dx} = 3x^2 e^{-y} \\ \implies & e^y \; dy = 3x^2 \; dx \\ \implies & \int e^y \; dy = \int 3x^2 \; dx \\ \implies & e^y = x^3 + C \\ \implies & y = \ln(x^3 + C). \end{aligned}$$

Check:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(x^3 + C)\right) = \frac{3x^2}{x^3 + C} = 3x^2 \frac{1}{x^3 + C}.$$

Recall $e^{\ln(a)} = a$, using this, we can write

$$\frac{dy}{dx} = 3x^2 e^{\ln\left(\frac{1}{x^3+C}\right)} = 3x^2 e^{-\ln(x^3+C)} = 3x^2 e^{-y}.$$

Now we apply the initial condition,

$$y(0) = \ln(C) = 2 \quad \implies \quad C = e^2,$$

so we have the final solution

$$y(x) = \ln(x^3 + e^2).$$

4.2.2 Integrating factors

Homogenous first order linear ODEs can be solved by seperating the variables. Non-homogenous ODEs cannot. In this section we will look at how to solve non-homogenous first order linear ODEs.

A first order linear ODE looks like

$$\frac{dy}{dx} + g(x)y = f(x).$$

When $f(x)$ is not zero, the ODE is non-homogenous. First order non-homogenous ODEs can be solved using integrating factors.

Definition

The **integrating factor** of the ODE

$$\frac{dy}{dx} + g(x)y = f(x).$$

is

$$\exp\left(\int g(x)\ dx\right).$$

When finding the integrating factor we can ignore the constant of integration.

If we multiply each term by the integrating factor, it should make our equation easier to solve. To solve

$$\frac{dy}{dx} + g(x)y = f(x),$$

let

$$T(x) = \int g(x)\ dx,$$

so that the integrating factor is

$$e^{T(x)}.$$

Multiplying through by the integrating factor gives

$$e^{T(x)}\frac{dy}{dx} + e^{T(x)}g(x)y = e^{T(x)}f(x),$$

By the product rule, we can find the derivative of $e^{T(x)}y$:

$$\begin{aligned}
\frac{d}{dx}\left(e^{T(x)}y\right) &= e^{T(x)}\frac{dy}{dx} + y\frac{d}{dx}\left(e^{T(x)}\right) \\
&= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}\frac{d}{dx}\left(T(x)\right) \\
&= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}q(x)
\end{aligned}$$

This is exactly what we had on the left hand side of the ODE after multiplying by the integrating factor. Therefore:

$$\begin{aligned}
\frac{d}{dx}\left(e^{T(x)}y\right) &= e^{T(x)}f(x) \\
e^{T(x)}y &= \int e^{T(x)}f(x)\ dx \\
y &= \frac{\int e^{T(x)}f(x)\ dx}{e^{T(x)}}
\end{aligned}$$

If we know how to integrate $e^{T(x)}f(x)$, then we can use this method to solve the ODE.

Example

Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x.$$

[note that $f(x, y) = x - (y/x)$ can't be separated.]
The integrating factor is:

$$\begin{aligned} \exp\left(\int \frac{1}{x}\, dx\right) &= \exp(\ln x) \\ &= x \end{aligned}$$

Multiplying through by the integrating factor gives:

$$x\frac{dy}{dx} + y = x^2$$

Notice that:

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

Therefore:

$$\begin{aligned} \frac{d}{dx}(xy) &= x^2 \\ xy &= \int x^2\, dx \\ &= \frac{x^3}{3} + c \\ y &= \frac{x^2}{3} + \frac{c}{x} \end{aligned}$$

Example

$$\begin{aligned}
\frac{dy}{dx} + xy &= x \\
\exp\left(\int x\ dx\right)\frac{dy}{dx} + \exp\left(\int x\ dx\right) xy &= \exp\left(\int x\ dx\right) x \\
\exp\left(\frac{1}{2}x^2\right)\frac{dy}{dx} + \exp\left(\frac{1}{2}x^2\right) xy &= \exp\left(\frac{1}{2}x^2\right) x \\
\frac{d}{dx}\left(\exp\left(\frac{1}{2}x^2\right) y\right) &= \exp\left(\frac{1}{2}x^2\right) x \\
\exp\left(\frac{1}{2}x^2\right) y &= \int \exp\left(\frac{1}{2}x^2\right) x\ dx \\
\exp\left(\frac{1}{2}x^2\right) y &= \exp\left(\frac{1}{2}x^2\right) \\
y &= \frac{\exp\left(\frac{1}{2}x^2\right) + c}{\exp\left(\frac{1}{2}x^2\right)} \\
y &= 1 + c\exp\left(-\frac{1}{2}x^2\right)
\end{aligned}$$

Or

$$y = 1 + ce^{-\frac{1}{2}x^2}$$

Example
Solve the initial-value problem

$$y' = y + x^2, \quad y(0) = 1.$$

First, find the general solution:

$$\begin{aligned}
\frac{dy}{dx} - y &= x^2 \\
\exp\left(\int -1\ dx\right)\frac{dy}{dx} - \exp\left(\int -1\ dx\right) y &= \exp\left(\int -1\ dx\right) x^2 \\
e^{-x}\frac{dy}{dx} - e^{-x}y &= e^{-x}x^2 \\
\frac{d}{dx}\left(e^{-x}y\right) &= e^{-x}x^2 \\
e^{-x}y &= \int e^{-x}x^2\ dx \\
&= -e^{-x}x^2 + \int 2xe^{-x}\ dx \\
&= -e^{-x}x^2 - 2xe^{-x} + \int 2e^{-x}\ dx \\
&= -e^{-x}x^2 - 2xe^{-x} - 2e^{-x} + c \\
y &= -x^2 - 2x - 2 + ce^{x}
\end{aligned}$$

Now we must use the initial condition to find c:

$$\begin{aligned} y &= -x^2 - 2x - 2 + ce^x \\ 1 &= -0^2 - 2 \cdot 0 - 2 + ce^0 \\ 1 &= -2 + c \\ 3 &= c \end{aligned}$$

Therefore the solution to the problem is

$$y = -x^2 - 2x - 2 + 3e^x.$$

4.3 Complementary functions and particular integrals

In the previous three examples, we gained the following results:

$$y = \frac{1}{3}x^2 + c \cdot .\frac{1}{x},$$

$$y = 1 + c \cdot e^{-\frac{1}{2}x^2},$$

$$y = -(x^2 + 2x + 2) + ce^x.$$

These examples have something very important in common, that is the solutions have the following form

$$y = f(x) + cg(x),$$

with explicit functions f and g.

Definition
When $y = f(x) + cg(x)$ is the solution of an ODE, f is called the **particular integral** (P.I.) and g is called the **complementary function** (C.F.).

We can use particular integrals and complementary functions to help solve ODEs if we notice that:

1. The complementary function (g) is the solution of the homogenous ODE.
2. The particular integral (f) is any solution of the non-homogenous ODE.

Example
We will use complementary functions and particular integrals to solve

$$y' + \lambda y = p(x), \quad \lambda \text{ is constant.}$$

We know that the general solution is

$$y(x) = \underbrace{f(x)}_{\text{particular integral (P.I.)}} + \underbrace{Cg(x)}_{\text{complementary function (C.F.)}},$$

where

$$f' + \lambda f = p(x),$$
$$g' + \lambda g = 0,$$

and C is the constant of integration.
We start by finding g. We need to solve

$$g' + \lambda g = 0.$$

The solution of this is

$$g = Ce^{-\lambda x}.$$

[This can be found be separating the variables or by inspection.]
Therefore, the general solition to $y' + \lambda y = p(x)$ is

$$y = f(x) + Ce^{-\lambda x},$$

where f is a particular integral.

The value of f depends on p. To find f, we make a guess at what it might look like then see if we are right.

Example: $p(x) = x$
Consider the differential equation

$$y' + y = x,$$

so we have

$$\lambda = 1, \quad p(x) = x.$$

The solution of this ODE will be of the form

$$y = f(x) + Ce^{-x}.$$

We can guess that f should be a polynomial with a degree of one, since $p(x) = x$. So we try the most general first order polynomial, $f(x) = ax + b$, and so $f'(x) = a$. Substituting $y = f(x)$ into the differential equation we have that

$$\begin{aligned} f' + f &= a + ax + b \\ &= ax + (a + b). \end{aligned}$$

And so

$$x \equiv ax + (a + b).$$

Thus, comparing coefficients from the LHS and RHS we must have that

$$\left.\begin{aligned} a &= 1 \\ a + b &= 0 \end{aligned}\right\} \quad \Longrightarrow \quad \left.\begin{aligned} a &= 1 \\ b &= -1 \end{aligned}\right\} \quad \Longrightarrow \quad f(x) = x - 1,$$

so $f(x) = x - 1$ is the particular integral. Therefore, the general solution to the original equation is

$$y(x) = x - 1 + Ce^{-x}.$$

When we solve higher order linear ODEs, we use a similar method:]

1. We solve the homogenous equation to find the complementary function.
2. We guess the form of the particular integral then try it.

4.4 Second order differential equations

In this section, we will learn how to solve second order ODEs with constant coefficients. These are ODEs of the form

$$y'' + Ay' + By = f(x)$$

or

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = f(x)$$

where $A, B \in \mathbb{R}$.

To solve equations like this, we look for particular integrals and complementary functions to make solutions of the form:

$$y(x) = \underbrace{f(x)}_{\text{particular integral (P.I.)}} + \underbrace{Cg(x)}_{\text{complementary function (C.F.)}},$$

4.4.1 Finding complementary functions

In this section we will be looking for solutions of the homogenous ODE

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0.$$

These will be the complementary functions of the non-homogenous ODE.

Definition
Two functions, y_1 and y_2 are **independent** if one is not a multiple of the other; otherwise, they are said to be **dependent**.

Example
$y_1(x) = 1$ and $y_2(x) = x$ are independent.
$y_1(x) = x$ and $y_2(x) = 3x$ are dependent because $y_2(x) = 3y_1(x)$.
$y_1(x) = e^{ax}$ and $y_2(x) = e^{bx}$ $(a \neq b)$ are independent.

Theorem

If $y_1(x)$ and $y_2(x)$ are independent and they are the solutions to

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

then the complementary function of this ODE is

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants.
In other words, if you have two independent solutions to the homogenous equation, then you can represent any other solution to the homogenous equation in terms of these two solutions.

Therefore our aim is to find two independent solutions to

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

If we look for solutions of the form

$$y = e^{\lambda x},$$

then

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x},$$

and so

$$\begin{aligned} y'' + Ay' + By &= \lambda^2 e^{\lambda x} + A\lambda e^{\lambda x} + Be^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 + A\lambda + B). \end{aligned}$$

Therefore

$$e^{\lambda x}(\lambda^2 + A\lambda + B) \equiv 0.$$

This will be zero when

$$\lambda^2 + A\lambda + B = 0$$

Definition

$$\lambda^2 + A\lambda + B = 0$$

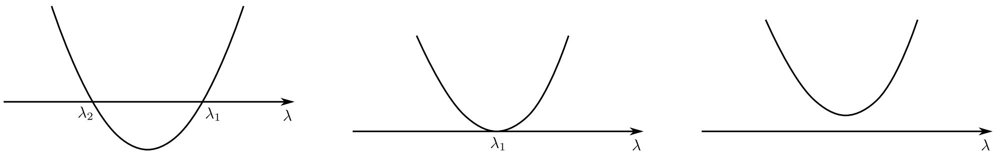
is called the **characteristic equation** or **auxiliary equation** of the ODE

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0.$$

In general, we know that the roots of the auxiliary equation are given by

$$\lambda = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

There are three cases, depending on the value of $\Delta = A^2 - 4B$.



(a) Two roots (b) One root (c) No roots (complex)

Figure 4.1: Different options for the curve $y = \lambda^2 + A\lambda + B$ when solving the equation $\lambda^2 + A\lambda + B = 0$.

Case 1: $\Delta > 0$

If $\Delta > 0$, we have two distinct real roots:

$$\lambda_1 = \frac{-r + \sqrt{\Delta}}{2} \quad \text{and} \quad \lambda_2 = \frac{-r - \sqrt{\Delta}}{2}$$

The general solution to the homogenous equation is

$$g(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

Example
Consider the second order differential equation

$$y'' + 3y' + 2y = 0.$$

The auxiliary equation is

$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda + 1)(\lambda + 2) = 0.$$

This has two roots:

$$\lambda_1 = -1, \quad \lambda_2 = -2.$$

Therefore we have two solutions e^{-x} and e^{-2x}, and they are independent. So the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}.$$

Case 2: $\Delta = 0$

If $\Delta = 0$, the auxiliary equation has one root which is real, given by

$$\lambda_1 = \frac{A}{B}.$$

So $e^{\lambda_1 x}$ is one solution to the homogenous equation. We need to find another solution, which is independent of $e^{\lambda_1 x}$. So we try

$$y_2(x) = x e^{\lambda_1 x}.$$

$$y' = e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x}$$

$$\begin{aligned} y'' &= \lambda_1 e^{\lambda_1 x} + \lambda_1 e^{\lambda_1 x} + \lambda_1^2 x e^{\lambda_1 x} \\ &= \lambda_1^2 x e^{\lambda_1 x} + 2\lambda_1 e^{\lambda_1 x}. \end{aligned}$$

Therefore

$$\begin{aligned} y'' + Ay' + By &= \lambda_1^2 x e^{\lambda_1 x} + 2\lambda_1 e^{\lambda_1 x} + A e^{\lambda_1 x} + A\lambda_1 x e^{\lambda_1 x} + B x e^{\lambda_1 x} \\ &= \underbrace{(\lambda_1^2 + A\lambda_1 + B)}_{=0} x e^{\lambda_1 x} + \underbrace{(2\lambda_1 + A)}_{=0} e^{\lambda_1 x} \\ &= 0. \end{aligned}$$

Therefore, $xe^{\lambda_1 x}$ is the second independent solution. Therefore, the general solution of the homogenous equation is

$$g(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

or

$$g(x) = (c_1 + c_2 x) e^{\lambda_1 x}.$$

Example
Consider the differential equation

$$y'' + 2y' + y = 0.$$

The auxiliary equation is:

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0$$

$$\lambda_1 = -1$$

λ_1 is the only root.
$e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are two independent solutions, so the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

or

$$y(x) = e^{-x}(c_1 + c_2 x).$$

Case 3: $\Delta < 0$

If $\Delta < 0$, there is no real solution to the auxiliary equation. But it is still possible to find two independent solutions, which give the general solution

$$g(x) = e^{\alpha x} \left(c_1 \cos \beta x + c_2 \sin \beta x \right),$$

where

$$\alpha = -\frac{A}{2}$$

and

$$\beta = \frac{\sqrt{4B - A^2}}{2}.$$

Complex numbers can be used to show that these are the solutions.[1]

Example
Consider the differential equation

$$y'' - 6y' + 13y = 0.$$

The auxiliary equation is

$$\lambda^2 - 6\lambda + 13 = 0,$$

which has

$$A = -6, \quad B = 13, \quad \Delta = A^2 - Bs = 36 - 52 = -16 < 0,$$

i.e. it has complex roots. So we have

$$\alpha = -\frac{-6}{2} = 3, \quad \beta = \frac{1}{2}\sqrt{4 \cdot 13 - (-6)^2} = 2,$$

therefore $e^{3x} \cos 2x$ and $e^{3x} \sin 2x$ are two independent solutions, so

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x),$$

is the general solution.

4.4.2 Finding a particular integral

The particular integral depends on the function $p(x)$. We only consider three categories of $p(x)$:

1. polynomials
2. trigonometric functions
3. exponential functions

p is a polynomial

When p is a polynomial, we guess that the particular integral will be a polynomial of the same order.

Example

[1] Details of how this is done can be found on Moodle or at `http://www.mscroggs.co.uk/6103`.

Find the general solution to the differential equation

$$y'' + 2y' + y = x^2.$$

Recall, the general solution takes the form $y = f(x) + g(x)$. Using the method in the previous section, we know that the C.F. is

$$g(x) = c_1 e^{-x} + c_2 x e^{-x}$$

or

$$g(x) = (c_1 + c_2 x)e^{-x}.$$

Next, we must find the particular integral (P.I.), we try

$$f(x) = ax^2 + bx + x.$$

We find

$$f'(x) = 2ax + b, \quad f''(x) = 2a.$$

Substituting into the differential equation gives

$$\begin{aligned} f'' + 2f' + f &= 2a + 2(2ax + b) + ax^2 + bx + c \\ &= ax^2 + (4a + b)x + 2a + 2b + c \\ &\equiv x^2. \end{aligned}$$

Comparing coefficients between the LHS and the RHS we have

$$\left.\begin{array}{l} a = 1 \\ 4a + b = 0 \\ 2a + 2b + c = 0 \end{array}\right\} \implies \left.\begin{array}{l} a = 1 \\ b = -4 \\ c = 6 \end{array}\right\} \implies f(x) = x^2 - 4x + 6,$$

Finally, we can write the general solution as

$$y(x) = x^2 - 4x + 6 + (c_1 + c_2 x)e^{-x}.$$

p is a trigonometric function

If p is a sin or cos, we guess that the particular integral will involve sin and cos.

Example
Solve the following initial-value problem:

$$y'' - 2y' + y = \sin x, \quad y(0) = -2, \quad y'(0) = 2.$$

[Notice that we have two boundary conditions here because second order differential equations have two constants of integration to be found.]
The C.F. for this problem is

$$g(x) = (c_1 + c_2 x)e^{x}.$$

To find the P.I. we try
$$f = a\sin x + b\cos x.$$
We find
$$f' = a\cos x - b\sin x, \quad f'' = -a\sin x - b\cos x.$$
Substituting into the differential equation we have

$$\begin{aligned} f'' - 2f' + f &= -a\sin x - b\cos x - 2a\cos x + 2b\sin x + a\sin x + b\cos x \\ &= (-a + 2b + a)\sin x + (-b - 2a + b)\cos x \\ &= 2b\sin x - 2a\cos x \\ &\equiv \sin x. \end{aligned}$$

Comparing coefficients, we have

$$a = 0, \quad b = \frac{1}{2} \quad \Longrightarrow \quad f = \frac{1}{2}\cos x.$$

Therefore the general solution to the initial-value problem is

$$y(x) = \frac{1}{2}\cos x + (c_1 + c_2 x)e^x.$$

In order to find the unknown constants c_1 and c_2 using the boundary conditions, we need to find $y'(x)$, so we differentiate the above to give:

$$\begin{aligned} y'(x) &= -\frac{1}{2}\sin x + c_2 e^x + (c_1 + c_2 x)e^x \\ &= -\frac{1}{2}\sin x + (c_1 + c_2 + c_2 x)e^x \end{aligned}$$

The boundary conditions give:

$$\begin{aligned} y(0) &= \frac{1}{2}\cos 0 + e^0(c_1 + c_2 \cdot 0) \\ &= \frac{1}{2} + c_1 = -2 \\ y'(0) &= -\frac{1}{2}\sin 0 + e^0(c_1 + c_2 + c_2 \cdot 0) \\ &= c_1 + c_2 = 2 \end{aligned}$$

Thus, we have the constants

$$c_1 = -\frac{5}{2}, \quad c_2 = 2 - c_1 = 2 + 52 = \frac{9}{2}.$$

Finally, the solution to the initial value problem is

$$y(x) = \frac{1}{2}\cos x + \frac{1}{2}e^x(9x - 5).$$

p is an exponential function

If p is an exponential function then we guess that the particular integral will be an exponential function.

Example
Find the general solution of the following differential equation,

$$y'' + 4y' + 3y = 5e^{4x}.$$

The auxiliary equation is

$$\lambda^2 + 4\lambda + 3 = 0 \qquad \Longleftrightarrow \qquad (\lambda+1)(\lambda+3) = 0.$$

Hence, this has two distinct real roots, namely $\lambda_1 = -1$, $\lambda_2 = -3$. So the C.F. (from the homogeneous equation) is given by

$$g(x) = c_1e^{-x} + c_2e^{-3x}.$$

To find the P.I. we try

$$f = ae^{4x},$$

since $p(x) = 5e^4x$. Differentiating, we have

$$f' = 4a^{4x}, \quad f'' = 16a^{4x}.$$

Substituting into the differential equation we see that

$$\begin{aligned} f'' + 4f' + 3f &= 16ae^{4x} + 16ae^{4x} + 3ae^{4x} \\ &= 35ae^{4x} \\ &= 35ae^{4x} & \equiv 5e^{4x}. \end{aligned}$$

Therefore we have $a = 1/7$ and so the P.I. is

$$f = \frac{1}{7}e^{4x}.$$

Finally, the general solution is

$$y(x) = \frac{1}{7}e^{4x} + c_1e^{-x} + c_2e^{-3x}.$$

Example
Solve the initial-value problem given by

$$y'' + 4y' + 3y = e^{-x}, \quad y(0) = 0, \quad y'(0) = 0.$$

We know from the example above that the C.F. is

$$c_1e^{-x} + c_2e^{-3x},$$

Now let us find the P.I., if we try $f = ae^{-x}$, we know that it wouldn't work since ae^{-x} is actually a solution to the homogeneous equation $y''+4y'+3y = 0$. Therefore, we try

$$f = axe^{-x},$$

thus

$$f' = ae^{-x} - axe^{-x}, \quad \text{and} \quad f'' = -2ae^{-x} + axe^{-x}.$$

Substituting into the differential equation we have

$$\begin{aligned} f'' + 4f' + 3f &= -2ae^{-x} + axe^{-x} + 4ae^{-x} - 4axe^{-x} + 3axe^{-x} \\ &= 2ae^{-x} \\ &\equiv e^{-x}. \end{aligned}$$

Therefore we must have $a = 1/2$ and so the general solution to the differential equations is

$$y(x) = \frac{1}{2}xe^{-x} + c_1e^{-x} + c_2e^{-3x}.$$

Differentiating the general solution we have

$$y'(x) = \frac{1}{2}e^{-x} - \frac{1}{2}xe^{-x} - c_1e^{-x} - 3c_2e^{-3x}.$$

Putting $x = 0$, we have (from the initial conditions)

$$\left.\begin{array}{l} y(0) = c_1 + c_2 = 0 \\ y'(0) = \frac{1}{2} - c_1 - 3c_2 = 0 \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{l} c_1 = -\frac{1}{4} \\ c_2 = \frac{1}{4} \end{array},$$

thus

$$y(x) = \frac{1}{2}xe^{-x} - \frac{1}{4}e^{-x} + \frac{1}{4}e^{-3x},$$

is the solution to the initial-value problem.

Example
Find the general solution to the equation

$$y'' + 2y' + y = 2e^{-x}.$$

Find the C.F.: The auxiliary equation is

$$\begin{aligned} \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda + 1)^2 &= 0 \\ \lambda_1 &= -1 \end{aligned}$$

We have a repeated root so

$$g(x) = c_1e^{-x} + c_2xe^{-x}.$$

Here e^{-x} and xe^{-x} are two independent solutions to the homogeneous equation

$$y'' + 2y' + y = 0.$$

Find a P.I.: We have to try

$$f = ax^2e^{-x},$$

since e^{-x} and xe^{-x} can't be the solution to the original differential equation as they satisfy the homogeneous equation. The derivatives are:

$$f' = 2axe^{-x} - ax^2e^{-x}$$

$$\begin{aligned} f'' &= 2ae^{-x} - 2axe^{-x} - 2axe^{-x} + ax^2e^{-x} \\ &= axe^{-x} - 4axe^{-x} + 2ae^{-x} \end{aligned}$$

Substituting $y = f(x)$ into the differential equation, we have:

$$\begin{aligned} f'' + 2f' + f &= ax^2e^{-x} - 4axe^{-x} + 2ae^{-x} + 4axe^{-x} - 2ax^2e^{-x} + ax^2e^{-x} \\ &= 2ae^{-x} \\ 2ae^{-x} &\equiv 2e^{-x} \end{aligned}$$

Therefore $a = 1$. So finally, we have the general solution

$$y(x) = (c_1 + c_2x + x^2)e^{-x}.$$

4.5 Solving initial-value problems numerically

Most differential equations can not be solved analytically, so we try to solve them numerically.

4.5.1 Euler's method

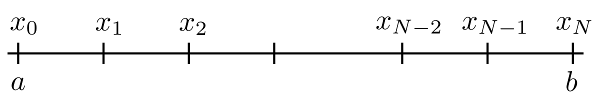
Suppose we have an initial-value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = y_0.$$

We want to find the solution $y(x)$ numerically on the interval $[a, b]$.

First we divide $[a, b]$ into N equal subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_k < \cdots < x_N = b.$$



Similarly to when we looked at the trapezium method:

$$x_k = a + kh, \quad h = \frac{b-a}{N} \quad \text{(step size)}, \quad k = 0, 1, \ldots, N.$$

If h is small, then the curve will be close to a straight line between x_k and x_{k+1}. The differential equation tells us that the gradient at x_k is $f(x_k, y_k)$. Therefore we approximate the curve between x_k and x_{k+1} by a straight line with gradient $f(x_k, y_k)$.

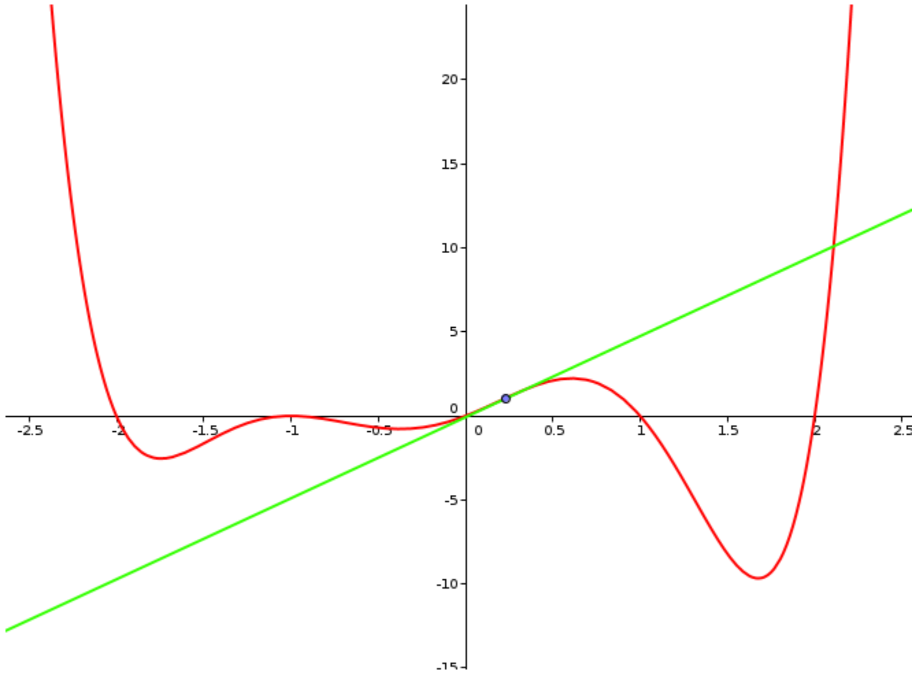


Figure 4.2: Near the point marked, the curve (red) is closely approximated by the tangent (green).

This gives us the following approximation for y_{k+1}:

$$y_{k+1} = y_k + hf(x_k, y_k)$$

We know that $y_0 = a$. We can then use this formula to approximate y_1, then use it again to find y_2, then y_3 and so on. The method of approximating y_k iteratively in this way is called **Euler's method**.

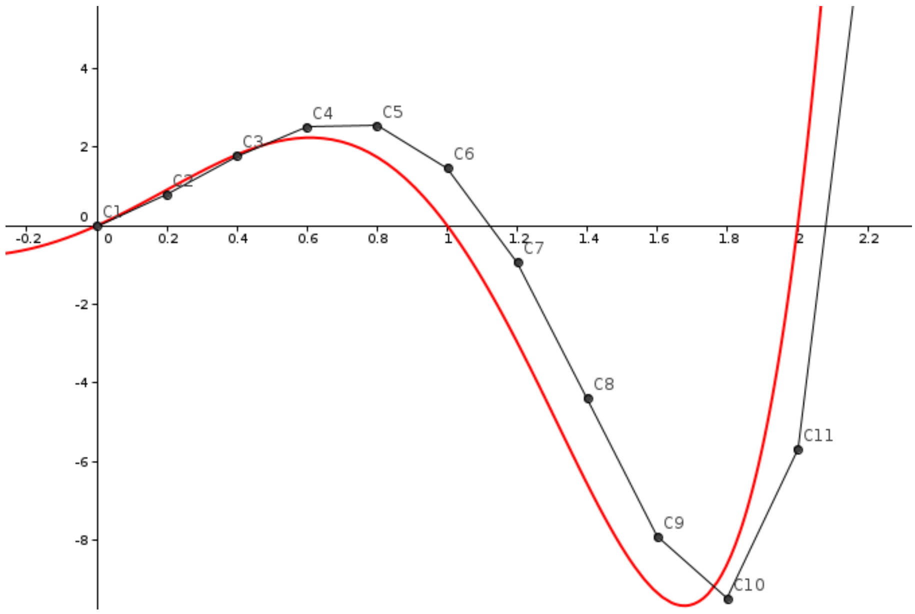


Figure 4.3: Using Euler's method to approximate a curve, starting at 0. at each point, a straight line is drawn with the gradient equal to that of the curve.

Example

Estimate $y(1)$, where $y(x)$ satisfies the initial-value problem:

$$\frac{dy}{dx} = y, \quad y(0) = 1.$$

We know the exact solution is

$$y(x) = e^x, \quad \Longrightarrow \quad y(1) = e \approx 2.71828.$$

Now we apply Euler's method to the problem. We have

$$f(x, y) = y.$$

First, we take $N = 5$, then $h = (1 - 0)/5 = 0.2$.

$$\begin{array}{cccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{array}$$

$$\begin{aligned}
y_0 &= y(0) = 1 \\
y_1 &= y_0 + hf(x_0, y_0) = 1 + 0.2 \times 1 = 1.2 \\
y_2 &= y_1 + hf(x_1, y_1) = 1.2 + 0.2 \times 1.2 = (1.2)^2 \\
y_3 &= y_2 + hf(x_2, y_2) = y_2 + hy_2 = y_2(1 + h) = (1.2)^2 \times 1.2 = (1.2)^3 \\
y_4 &= (1.2)^4 \\
y_5 &= (1.2)^5 \approx 2.48832.
\end{aligned}$$

Euler's method with 5 subintervals has given us the approximation

$$y(1) \approx 2.48832.$$

As we know the exact solution, we can look at the error:

$$\begin{aligned} \text{error} &= e - y_5 \\ &= 2.71828 - 2.48832 \\ &= 0.22996 \end{aligned}$$

Now, we double the number of subintervals: $N = 10$, $h = 0.1$ then we need 10 steps to reach $x_{10} = 1$.

$$y_{10} = (1.1)^{10} \approx 2.59374,$$

then we have

$$\text{error} = 2.71828 - 2.59374 = 0.12454.$$

For $N = 20$, $h = 0.05$ and so

$$y_{20} = (1.05)^{20} \approx 2.65330, \quad \text{error} = 0.0650.$$

For $N = 40$, $h = 0.025$ and so

$$y_{20} = (1.025)^{40} \approx 2.68506, \quad \text{error} = 0.0332.$$

As we increase the number of intervals, the value becomes a better approximation.

4.6 Applications of ODEs

4.6.1 Simple harmonic motion (SHM)

Pendulums, objects bouncing on springs and molecular vibration can, after some simplifying assumptions, all be described by the same ODE:

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

This is often written as:

$$\ddot{\theta} + \omega^2\theta = 0$$

Motion described by this equation is called **simple harmonic motion** (or **SHM**). It can be thought of in general as describing small oscillations.

Let's consider a pendulum and see why it can be described by SHM.

Example: A pendulum
An ideal pendulum consists of a weightless rod of length l attached at one end to a frictionless hinge and supporting a body of mass m at the other end. We describe the motion in terms of angle θ, made by the rod and the vertical.

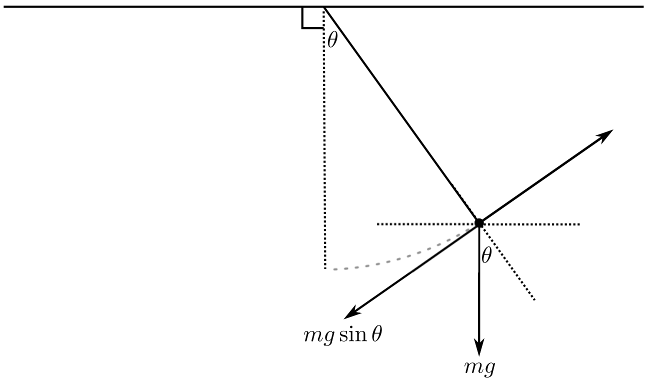


Figure 4.4: Sketch of a pendulum of length l with a mass m, displaying the forces acting on the mass resolved in the tangential direction relative to the motion.

Using Newton's second law of motion $F = ma$, we have the differential equation:

$$-mg\sin\theta = ml\ddot{\theta}$$

We re-write the equation as

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

This is a nonlinear equation, and we can not solve it analytically.
When θ is small, $\sin\theta \approx \theta$ and so:

$$\ddot{\theta} + \omega^2\theta = 0, \quad \omega = \sqrt{\frac{g}{l}}.$$

This is the equation for simple harmonic motion which we gave above. We can now solve this equation.
The auxilliary equation is

$$\begin{aligned}\lambda^2 + \omega^2 &= 0\\ \lambda^2 &= -\omega^2\\ \lambda &= \pm\omega\sqrt{-1}\end{aligned}$$

Therefore the solution of the ODE is:

$$\theta(t) = A\cos\omega t + B\sin\omega t.$$

If the pendulum is displaced by an angle θ_0 and released, then $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, so

$$\theta(0) = A = \theta_0, \quad \dot{\theta}(0) = B\omega = 0 \quad \Longrightarrow \quad B = 0,$$

therefore

$$\theta(t) = \theta_0\cos\omega t \quad \Longrightarrow \quad |\theta(t)| \leq \theta_0.$$

The example above claims that the object on the pendulum will continue to swing forever. In reality the pendulum will slow down due to air resistance.

In order for our pendulum to do this, we must add air resistance to the ODE.

Example: A pendulum with air resistance
As the pendulum is swinging, it will be subject to air resistance. The force due to air resistance will be proportional to the speed of the pendulum. This leads to the ODE

$$\ddot{\theta} + c\dot{\theta} + \frac{g}{l}\theta = 0,$$

where c is a constant.

We have learnt enough about ODEs that we can solve this equation, although we're not going to.

This is the end of the course. Let's finish with a limerick[2] :

$$\int_1^{\sqrt[3]{3}} v \cdot v \; dv \cos\left(\frac{3\pi}{9}\right) = \ln\left(\sqrt[3]{e}\right)$$

[2]Translation:

Integral v times v dv,
From 1 to the cube root of 3,
Times by the cosine,
Of three pi by 9,
Is log of the cube root of e.