**MATH6103 Differential & Integral Calculus**

**MATH6500 Elementary Mathematics for Engineers**

# Solutions to Problem Sheet 9

1) Solve the following:

a) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda + 1)(\lambda + 2) &= 0 \\ \lambda &= -1 \text{ or } -2 \end{aligned}$$

So the general solution is

$$y = Ae^{-x} + Be^{-2x}.$$

b) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 - 3\lambda + 2 &= 0 \\ (\lambda - 1)(\lambda - 2) &= 0 \\ \lambda &= 1 \text{ or } 2 \end{aligned}$$

So the general solution is

$$y = Ae^{x} + Be^{2x}.$$

c) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \sin x$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 - 3\lambda + 2 &= 0 \\ (\lambda - 1)(\lambda - 2) &= 0 \\ \lambda &= 1 \text{ or } 2 \end{aligned}$$

So the complementary function is

$$y = Ae^{x} + Be^{2x}.$$

For the particular integral, try $y = \alpha \sin x + \beta \cos x$.

$$\begin{aligned} y &= \alpha \sin x + \beta \cos x \\ y' &= \alpha \cos x - \beta \sin x \\ y'' &= -\alpha \sin x + -\beta \cos x \\ \sin x &= \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y \\ &= -\alpha \sin x + -\beta \cos x - 3(\alpha \cos x - \beta \sin x) + 2(\alpha \sin x + \beta \cos x) \\ &= -\alpha \sin x + -\beta \cos x - 3\alpha \cos x + 3\beta \sin x + 2\alpha \sin x + 2\beta \cos x \\ &= (\alpha + 3\beta) \sin x + (\beta - 3\alpha) \cos x \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} \alpha + 3\beta &= 1 \\ \beta - 3\alpha &= 0 \end{aligned}$$

These give:

$$\begin{aligned} \alpha &= \frac{1}{10} \\ \beta &= \frac{3}{10} \end{aligned}$$

So the particular integral is

$$y = \frac{1}{10} \sin x + \frac{3}{10} \cos x.$$

Therefore the general solution is

$$y = Ae^{x} + Be^{2x} + \frac{1}{10} \sin x + \frac{3}{10} \cos x.$$

d) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = x^2$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda + 1)(\lambda + 2) &= 0 \\ \lambda &= -1 \text{ or } -2 \end{aligned}$$

So the complementary function is

$$y = Ae^{-x} + Be^{-2x}.$$

For the particular integral, try $y = \alpha x^2 + \beta x + \gamma$.

$$\begin{aligned} y &= \alpha x^2 + \beta x + \gamma \\ y' &= 2\alpha x + \beta \\ y'' &= 2\alpha \\ x^2 &= \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y \\ &= 2\alpha + 3\left(2\alpha x + \beta\right) + 2\left(\alpha x^2 + \beta x + \gamma\right) \\ &= 2\alpha x^2 + \left(6\alpha + 2\beta\right)x + 3\beta + 2\gamma + 2\alpha \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} 2\alpha &= 1 \\ 6\alpha + 2\beta &= 0 \\ 3\beta + 2\gamma + 2\alpha &= 0 \end{aligned}$$

These give:

$$\begin{aligned} \alpha &= \frac{1}{2} \\ \beta &= \frac{-3}{2} \\ \gamma &= \frac{7}{4} \end{aligned}$$

So the particular integral is

$$y = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}.$$

Therefore the general solution is

$$y = Ae^{-x} + Be^{2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}.$$

e) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \sin x - \cos x$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 - 3\lambda + 2 &= 0 \\ \left(\lambda - 1\right)\left(\lambda - 2\right) &= 0 \\ \lambda &= 1 \text{ or } 2 \end{aligned}$$

So the complementary function is

$$y = Ae^{x} + Be^{2x}.$$

For the particular integral, try $y = \alpha \sin x + \beta \cos x$.

$$\begin{aligned}
y &= \alpha \sin x + \beta \cos x \\
y' &= \alpha \cos x - \beta \sin x \\
y'' &= -\alpha \sin x + -\beta \cos x \\
\sin x - \cos x &= \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y \\
&= -\alpha \sin x + -\beta \cos x - 3\left(\alpha \cos x - \beta \sin x\right) + 2\left(\alpha \sin x + \beta \cos x\right) \\
&= -\alpha \sin x + -\beta \cos x - 3\alpha \cos x + 3\beta \sin x + 2\alpha \sin x + 2\beta \cos x \\
&= (\alpha + 3\beta) \sin x + (\beta - 3\alpha) \cos x
\end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned}
\alpha + 3\beta &= 1 \\
\beta - 3\alpha &= -1
\end{aligned}$$

These give:

$$\begin{aligned}
\alpha &= \frac{1}{5} \\
\beta &= \frac{1}{15}
\end{aligned}$$

So the particular integral is

$$y = \frac{1}{5} \sin x + \frac{1}{15} \cos x.$$

Therefore the general solution is

$$y = Ae^{x} + Be^{2x} + \frac{1}{10} \sin x + \frac{3}{10} \cos x.$$

f) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x}$

Solve the auxiliary equation:

$$\begin{aligned}
\lambda^2 + 3\lambda + 2 &= 0 \\
(\lambda + 1)(\lambda + 2) &= 0 \\
\lambda &= -1 \text{ or } -2
\end{aligned}$$

So the complementary function is

$$y = Ae^{-x} + Be^{-2x}.$$

For the particular integral, try $y = \alpha e^{2x}$.

$$\begin{aligned} y &= \alpha e^{2x} \\ y' &= 2\alpha e^{2x} \\ y'' &= 4\alpha e^{2x} \\ e^{2x} &= \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y \\ &= 4\alpha e^{2x} + 6\alpha e^{2x} + 2\alpha e^{2x} \\ &= 12\alpha e^{2x} \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} 12\alpha &= 1 \\ \alpha &= \frac{1}{12} \end{aligned}$$

So the particular integral is

$$y = \frac{1}{12}e^{2x}.$$

Therefore the general solution is

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{12}e^{2x}.$$

g) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{2x}$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 - 3\lambda + 2 &= 0 \\ (\lambda - 1)(\lambda - 2) &= 0 \\ \lambda &= 1 \text{ or } 2 \end{aligned}$$

So the complementary function is

$$y = Ae^{x} + Be^{2x}.$$

For the particular integral, try $y = \alpha x e^{2x}$.

$$\begin{aligned} y &= \alpha x e^{2x} \\ y' &= 2\alpha x e^{2x} + \alpha e^{2x} \\ y'' &= 4\alpha x e^{2x} + 4\alpha e^{2x} \\ e^{2x} &= \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y \\ &= 4\alpha x e^{2x} + 4\alpha e^{2x} - 3\left(2\alpha x e^{2x} + \alpha e^{2x}\right) + 2\alpha x e^{2x} \\ &= 4\alpha x e^{2x} + 4\alpha e^{2x} - 6\alpha x e^{2x} + 3\alpha e^{2x} + 2\alpha x e^{2x} \\ &= 4\alpha e^{2x} + 3\alpha e^{2x} \\ &= 7\alpha e^{2x} \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} 7\alpha &= 1 \\ \alpha &= \frac{1}{7} \end{aligned}$$

So the particular integral is

$$y = \frac{1}{7} x e^{2x}.$$

Therefore the general solution is

$$y = Ae^{x} + Be^{2x} + \frac{1}{7} x e^{2x}.$$

h) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x} + \sin x$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda + 1)(\lambda + 2) &= 0 \\ \lambda &= -1 \text{ or } -2 \end{aligned}$$

So the complementary function is

$$y = Ae^{-x} + Be^{-2x}.$$

For the particular integral, try $y = \alpha e^{2x} + \beta \sin x + \gamma \cos x$.

$$\begin{aligned}
y &= \alpha e^{2x} + \beta \sin x + \gamma \cos x \\
y' &= 2\alpha e^{2x} + \beta \cos x - \gamma \sin x \\
y'' &= 4\alpha e^{2x} - \beta \sin x - \gamma \cos x \\
e^{2x} &= \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y \\
&= 4\alpha e^{2x} - \beta \sin x - \gamma \cos x + 6\alpha e^{2x} + 3\beta \cos x - 3\gamma \sin x + 2\alpha e^{2x} + 2\beta \sin x + 2\gamma \cos x \\
&= 12\alpha e^{2x} + (\beta - 3\gamma) \sin x + (\gamma + 3\beta) \cos x
\end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned}
12\alpha &= 1 \\
\beta - 3\gamma &= 1 \\
\gamma + 3\beta &= 0
\end{aligned}$$

These give:

$$\begin{aligned}
\alpha &= \frac{1}{12} \\
\beta &= \frac{1}{10} \\
\gamma &= \frac{1}{30}
\end{aligned}$$

So the particular integral is

$$y = \frac{1}{12}e^{2x} + \frac{1}{10}\sin x + \frac{1}{30}\cos x.$$

Therefore the general solution is

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{12}e^{2x} + \frac{1}{10}\sin x + \frac{1}{30}\cos x.$$

i) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 3x - 2$

Solve the auxiliary equation:

$$\begin{aligned}
\lambda^2 + 3\lambda + 2 &= 0 \\
(\lambda + 1)(\lambda + 2) &= 0 \\
\lambda &= -1 \text{ or } -2
\end{aligned}$$

So the complementary function is

$$y = Ae^{-x} + Be^{-2x}.$$

For the particular integral, try $y = \alpha x + \beta$.

$$\begin{aligned} y &= \alpha x + \beta \\ y' &= \alpha \\ y'' &= 0 \\ 3x - 2 &= \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y \\ &= 0 + 3\alpha + 2\alpha x + 2\beta \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} 2\alpha &= 3 \\ 3\alpha + 2\beta &= -1 \end{aligned}$$

These give:

$$\begin{aligned} \alpha &= \frac{2}{3} \\ \beta &= -\frac{3}{2} \end{aligned}$$

So the particular integral is

$$y = \frac{2}{3}x - \frac{3}{2}.$$

Therefore the general solution is

$$y = Ae^{-x} + Be^{-2x} + \frac{2}{3}x - \frac{3}{2}.$$

j) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 2\lambda - 3 &= 0 \\ (\lambda - 1)(\lambda + 3) &= 0 \\ \lambda &= 1 \text{ or } -3 \end{aligned}$$

So the general solution is

$$y = Ae^{x} + Be^{-3x}.$$

k) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = \sin x$

Solve the auxiliary equation:

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} \qquad = -2 \pm \frac{1}{2}\sqrt{-4} = -2 \pm i$$

So the complementary function is

$$y = e^{-2x}\left(A\sin x + B\cos x\right).$$

For the particular integral, try $y = \alpha \sin x + \beta \cos x$.

$$\begin{aligned}
y &= \alpha \sin x + \beta \cos x \\
y' &= \alpha \cos x - \beta \sin x \\
y'' &= -\alpha \sin x + -\beta \cos x \\
\sin x &= \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y \\
&= -\alpha \sin x + -\beta \cos x + 4\alpha \cos x - 4\beta \sin x + 5\alpha \sin x + 5\beta \cos x \\
&= (4\alpha + 4\beta)\cos x\,(4\alpha - 4\beta)\sin x
\end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned}
4\alpha + 4\beta &= 0 \\
4\alpha - 4\beta &= 1
\end{aligned}$$

These give:

$$\begin{aligned}
\alpha &= \frac{1}{8} \\
\beta &= -\frac{1}{8}
\end{aligned}$$

So the particular integral is

$$y = \frac{1}{8}\sin x - \frac{1}{8}\cos x.$$

Therefore the general solution is

$$y = e^{-2x}\left(A\sin x + B\cos x\right) + \frac{1}{8}\sin x - \frac{1}{8}\cos x.$$

l) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = 0$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 4\lambda &= 0 \\ \lambda(\lambda + 4) &= 0 \\ \lambda &= 0 \text{ or } -4 \end{aligned}$$

So the general solution is

$$y = A + Be^{-4x}.$$

m) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = x + 3$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 4\lambda + 4 &= 0 \\ (\lambda + 2)^2 &= 0 \\ \lambda &= -2 \end{aligned}$$

So the complementary function is

$$y = (Ax + B)\, e^{-2x}.$$

For the particular integral, try $y = \alpha x + \beta$.

$$\begin{aligned} y &= \alpha x + \beta \\ y' &= \alpha \\ y'' &= 0 \\ x + 3 &= \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y \\ &= 0 + 4\alpha + 4\alpha x + 4\beta \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} 4\alpha &= 1 \\ 4\alpha + 4\beta &= 3 \end{aligned}$$

These give:

$$\begin{aligned} \alpha &= \frac{1}{4} \\ \beta &= -\frac{1}{2} \end{aligned}$$

So the particular integral is

$$y = \frac{1}{4}x + \frac{1}{2}.$$

Therefore the general solution is

$$y = (Ax + B)\, e^{-2x} + \frac{1}{4}x + \frac{1}{2}.$$

n) $x\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} = x^2$

$$\begin{aligned} x\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} &= x^2 \\ \frac{d^2y}{dx^2} + 6\frac{dy}{dx} &= x \end{aligned}$$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 6\lambda &= 0 \\ \lambda(\lambda + 6) &= 0 \\ \lambda &= 0 \text{ or } -6 \end{aligned}$$

So the complementary function is

$$y = A + Be^{-6x}.$$

For the particular integral, try $y = \alpha x + \beta x^2$. [As there is a constant in the C.F., multiply the normal guess by $x$.]

$$\begin{aligned} y &= \alpha x + \beta x^2 \\ y' &= \alpha + 2\beta x \\ y'' &= 2\beta \\ x &= \frac{d^2y}{dx^2} + 6\frac{dy}{dx} \\ &= 2\beta + 6\alpha + 12\beta x \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} 12\beta &= 1 \\ 6\alpha + 2\beta &= 0 \end{aligned}$$

These give:

$$\begin{aligned} \alpha &= \frac{1}{12} \\ \beta &= -\frac{1}{36} \end{aligned}$$

So the particular integral is

$$y = \frac{1}{12}x - \frac{1}{36}x^2.$$

Therefore the general solution is

$$y = A + Be^{-6x} + \frac{1}{12}x - \frac{1}{36}x^2.$$

o) $4\frac{d^2y}{dx^2} - y = 8x$

Solve the auxiliary equation:

$$\begin{aligned} 4\lambda^2 - 1 &= 0 \\ \lambda^2 &= \frac{1}{4} \\ \lambda &= \pm\frac{1}{2} \end{aligned}$$

So the complementary function is

$$y = Ae^{\frac{x}{2}} + Be^{-\frac{x}{2}}.$$

For the particular integral, try $y = \alpha x + \beta$.

$$\begin{aligned} y &= \alpha x + \beta \\ y' &= \alpha \\ y'' &= 0 \\ 8x &= 4\frac{d^2y}{dx^2} - y \\ &= 0 - \alpha x - \beta \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} \alpha &= -8 \\ \beta &= 0 \end{aligned}$$

So the particular integral is

$$y = -8x.$$

Therefore the general solution is

$$y = Ae^{\frac{x}{2}} + Be^{-\frac{x}{2}} - 8x.$$

p) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^x$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda + 1)^2 &= 0 \\ \lambda &= -1 \end{aligned}$$

So the complementary function is

$$y = (Ax + B)\, e^{-x}.$$

For the particular integral, try $y = \alpha e^x$.

$$\begin{aligned} y &= \alpha e^x \\ y' &= \alpha e^x \\ y'' &= \alpha e^x \\ e^x &= \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y \\ &= 4\alpha e^x \end{aligned}$$

Comparing coefficients gives:

$$\alpha = \frac{1}{4}$$

So the particular integral is

$$y = \frac{1}{4}e^x.$$

Therefore the general solution is

$$y = (Ax + B)\, e^{-x} + \frac{1}{4}e^x.$$

q) $\dfrac{d^2y}{dx^2} + y = x$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 1 &= 0 \\ \lambda^2 &= -1 \\ \lambda &= \pm i \end{aligned}$$

So the complementary function is

$$y = A\sin x + B\cos x.$$

For the particular integral, try $y = \alpha x + \beta$.

$$\begin{aligned} y &= \alpha x + \beta \\ y' &= \alpha \\ y'' &= 0 \\ x &= \frac{d^2y}{dx^2} + y \\ &= \alpha x + \beta \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} \alpha &= 1 \\ \beta &= 0 \end{aligned}$$

So the particular integral is

$$y = x.$$

Therefore the general solution is

$$y = A\sin x + B\cos x + x.$$

r) $\dfrac{d^2y}{dx^2} + y = \sin x$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^2 + 1 &= 0 \\ \lambda^2 &= -1 \\ \lambda &= \pm i \end{aligned}$$

So the complementary function is

$$y = A \sin x + B \cos x.$$

For the particular integral, try $y = \alpha x \sin x + \beta x \cos x$.

$$\begin{aligned} y &= \alpha x \sin x + \beta x \cos x \\ y' &= \alpha \sin x + \alpha x \cos x - \beta x \sin x + \beta \cos x \\ y'' &= \alpha \cos x + \alpha \cos x - \alpha x \sin x - \beta x \cos x - \beta \sin x - \beta \sin x \end{aligned}$$

$$\begin{aligned} \sin x &= \frac{d^2y}{dx^2} + y \\ &= \alpha \cos x + \alpha \cos x - \beta \sin x - \beta \sin x \\ &= \alpha \cos x + \alpha \cos x - \alpha x \sin x - \beta x \cos x - \beta \sin x - \beta \sin x + \alpha x \sin x + \beta x \cos x \\ &= 2\alpha \cos x - 2\beta \sin x \end{aligned}$$

Comparing coefficients gives:

$$\begin{aligned} \alpha &= 0 \\ \beta &= -\frac{1}{2} \end{aligned}$$

So the particular integral is

$$y = -\frac{1}{2} x \cos x.$$

Therefore the general solution is

$$y = A \sin x + B \cos x - \frac{1}{2} x \cos x.$$

**Challenge**: Adapt the method we have learned to solve this ODE:

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^{3x}$$

Solve the auxiliary equation:

$$\begin{aligned} \lambda^3 + 2\lambda^2 - \lambda - 2 &= 0 \\ (\lambda - 1)(\lambda + 2)(\lambda + 1) &= 0 \\ \lambda &= 1 \text{ or } -2 \text{ or } -1 \end{aligned}$$

So the complementary function is

$$y = Ae^{x} + Be^{-2x} + Ce^{-x}.$$

For the particular integral, try $y = \alpha e^{3x}$.

$$\begin{aligned}
y &= \alpha e^{3x} \\
y' &= 3\alpha e^{3x} \\
y'' &= 9\alpha e^{3x} \\
y''' &= 27\alpha e^{3x} \\
e^{3x} &= \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y \\
&= 27\alpha e^{3x} + 18\alpha e^{3x} - 3\alpha e^{3x} - 2\alpha e^{3x} \\
&= 40\alpha e^{3x}
\end{aligned}$$

Comparing coefficients gives:

$$\alpha = \frac{1}{40}$$

So the particular integral is

$$y = \frac{1}{40}e^{3x}.$$

Therefore the general solution is

$$y = Ae^{x} + Be^{-2x} + Ce^{-x} + \frac{1}{40}e^{3x}.$$

2) [*Calculator allowed*] Use Euler's method with $h = \frac{1}{2}$ to estimate $y(1)$ when

$$\frac{dy}{dx} = \sin y + x^2$$

and

$$y(0) = 0.$$

$$
\begin{aligned}
x_0 &= 0 \\
x_1 &= \frac{1}{2} \\
x_2 &= 1 \\
y_0 &= 0 \\
y_1 &= y_0 + h\left(\sin y_0 + x_0^2\right) \\
&= 0 + \frac{1}{2}\left(\sin 0 + 0^2\right) \\
&= 0 \\
y_2 &= y_1 + h\left(\sin y_1 + x_1^2\right) \\
&= 0 + \frac{1}{2}\left(\sin 0 + \left(\frac{1}{2}\right)^2\right) \\
&= \frac{1}{8}
\end{aligned}
$$

3) [*Calculator allowed*] Use Euler's method with $h = \frac{1}{5}$ to estimate $y(1)$ when

$$\frac{dy}{dx} = \sin y + x^2$$

and

$$y(0) = 0.$$

$$
\begin{aligned}
x_0 &= 0 \\
x_1 &= \frac{1}{5} \\
x_2 &= \frac{2}{5} \\
x_3 &= \frac{3}{5} \\
x_4 &= \frac{4}{5} \\
x_5 &= 1 \\
y_0 &= 0 \\
y_1 &= y_0 + h\left(\sin y_0 + x_0^2\right) \\
&= 0 + \frac{1}{5}\left(\sin 0 + 0^2\right) \\
&= 0 \\
y_2 &= y_1 + h\left(\sin y_1 + x_1^2\right) \\
&= 0 + \frac{1}{5}\left(\sin 0 + \left(\frac{1}{5}\right)^2\right) \\
&= 0.008 \\
y_3 &= y_2 + h\left(\sin y_2 + x_2^2\right) \\
&= 0.008 + \frac{1}{5}\left(\sin 0.008 + \left(\frac{2}{5}\right)^2\right) \\
&= 0.0416 \\
y_4 &= y_3 + h\left(\sin y_3 + x_3^2\right) \\
&= 0.0416 + \frac{1}{5}\left(\sin 0.0416 + \left(\frac{3}{5}\right)^2\right) \\
&= 0.1219 \\
y_5 &= y_4 + h\left(\sin y_4 + x_4^2\right) \\
&= 0.1219 + \frac{1}{5}\left(\sin 0.1219 + \left(\frac{4}{5}\right)^2\right) \\
&= 0.27421
\end{aligned}
$$